Mathematics for Business Decisions, Part I

Homework Set 10: Continuous Random Variables

All problems use continuous random variables.

The three most important relations for both discrete and continuous random variables are:

1. \( P(X \leq x) = F_X(x) \) (the “\( X \) is at most \( x \)” case for both discrete and continuous R.V.s)
2. \( P(X > x) = 1 - F_X(x) \) (the “\( X \) is at least \( x \)” case for continuous R.V.s)
3. \( P(c < x \leq d) = F_X(d) - F_X(c) \) (the “\( X \) is between” case for both discrete and continuous)

These are the three fundamental cases.

Comments:

Property 1 is just the definition of the cumulative distribution function.

Property 2 follows from the fact that \( P(X > x) = 1 - P(X \leq x) \). The probability \( P(X > x) \) is some times expresses as the survival function \( S(x) = P(X > x) \) since if we think of \( X \) as time until death or failure, then \( S(x) = P(X > x) \) is the probability that the individual or part survives to at least time \( x \).

Property 3 is more general than it first appears to be. For example, for a continuous random variable the probability that \( X \) takes on any particular value \( x \) is zero. That is, \( P(X = x) = 0 \) for any \( x \). It then follow that

\[
P(c < x \leq d) = P(c < x < d) = P(c \leq x < d) = P(c \leq x \leq d) = F_X(d) - F_X(c).
\]

Thus, given the three properties above and the formulas for the cumulative distribution function \( F_X(x) \), you can solve any of the following problems by manipulating the definitions to find the formula for the desired probability.

We now derive the cumulative distribution for the uniform and exponential distributions.
Derivation of the formulas for random variables with a uniform distribution:

If $X$ has a uniform distribution, then

**The formula for case 1:** $P(X \leq x) = \frac{x-a}{b-a}$.  
(Formula u1)

**Proof:** This comes straight from determining the area under the density curve $y = f_X(x)$, which is simply the area of the rectangle with base $(x-a)$ and height $\frac{1}{b-a}$. Thus $F_X(x) = \frac{x-a}{b-a}$.

**The formula for case 2:** $P(X > x) = \frac{b-x}{b-a}$  
(Formula u2)

**Proof:** $P(X > x) = 1 - P(X \leq x) = 1 - \frac{x-a}{b-a} = \frac{b-a}{b-a} - \frac{x-a}{b-a} = \frac{b-x}{b-a}$.

**The formula for case 3:** $P(c \leq X \leq d) = \frac{d-c}{b-a}$.  
(Formula u3)

**Proof:** $P(c \leq X \leq d) = F_X(d) - F_X(c) = \frac{d-a}{b-a} - \frac{c-a}{b-a} = \frac{d-c}{b-a}$

Derivation of the formulas for random variables with an exponential distribution:

If $X$ has an exponential distribution, then

**The formula for case 1:** $P(X \leq x) = 1 - \text{Exp} \left(-\frac{x}{\alpha}\right)$.  
(Formula e1)

**Proof:** (The at most case) The proof of this requires calculus. I will give it here, but just ignore it if you have never had calculus. You can simply accept Formula 1 as a magic formula. Recall that the probability density function for an exponential distribution is given by

$$f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{\alpha} e^{-\frac{x}{\alpha}} & \text{if } x \geq 0 \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t) \, dt = \int_{0}^{\frac{x}{\alpha}} e^{-\frac{t}{\alpha}} \, dt = -e^{-\frac{t}{\alpha}} \Bigg|_{0}^{\frac{x}{\alpha}} = e^{\frac{x}{\alpha}} - e^{0} = 1 - e^{-\frac{x}{\alpha}}$$
The formula for case 2: \( P(X > x) = \text{Exp} \left( -\frac{x}{\alpha} \right) \)  
(the survival function -Formula e2)

Proof: (The at least case) For any range value \( x \) we have

\[
P(X > x) = 1 - P(X \leq x) \quad \text{(by complementation)}
= 1 - F_X(x) \quad \text{(by definition of the cumulative distribution)}
= \text{Exp} \left( -\frac{x}{\alpha} \right) \quad \text{(by definition of the exponential function: see class notes)}
\]

NOTE: \( P(X \geq x) = P(X = x) + P(X > x) = P(X > x) \) (since \( X \) is a continuous random variable \( P(X = x) = 0 \) for any value of \( x \)). Thus, for a continuous random variable the at least case (i.e., \( P(X \geq x) \)) is equivalent strict inequality associated with the survival function, namely, the \( P(X > x) \) case.

The formula for case 3: \( P(c \leq X \leq d) = \text{Exp} \left( -\frac{c}{\alpha} \right) - \text{Exp} \left( -\frac{d}{\alpha} \right) \)  
(Formula e3)

Proof: (the between case)

\[
P(c \leq X \leq d) = F_X(d) - F_X(c)
= \left[ 1 - \text{Exp} \left( -\frac{d}{\alpha} \right) \right] - \left[ 1 - \text{Exp} \left( -\frac{c}{\alpha} \right) \right]
= \text{Exp} \left( -\frac{c}{\alpha} \right) - \text{Exp} \left( -\frac{d}{\alpha} \right)
\]
1-7. Let \( X \) be a continuous random variable with a uniform distribution. If \( a = 1 \) and \( b = 7 \), what are

1. \( P(X = 2) \)
2. \( P(X \leq 2) \)
3. \( P(X > 2) \)
4. \( P(1 < X \leq 2) \)
5. \( P(1 \leq X \leq 2) \)
6. \( P(3 < X < 5) \)
7. \( P(6 < X) \)

**Recall:** For a uniform distribution: 
\[
F_X(x) = P(X \leq x) = \frac{x-a}{b-a}
\]
(cumulative distribution).

**Solutions (1-7):** Using \( a = 1 \) and \( b = 7 \) in this cumulative distribution formula yields

\[
F_X(x) = P(X \leq x) = \frac{x-1}{7-1} = \frac{x-1}{6}
\]

1. \( P(X = 2) = 0 \) (because the probability \( P(X = x) = 0 \) for every continuous random variable).

2. \( P(X \leq 2) = F_X(2) = \frac{2-1}{6} = \frac{1}{6} \)

3. \( P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - \frac{1}{6} = \frac{5}{6} \)

4. \( P(1 < X \leq 2) = F_X(2) - F_X(1) = \frac{2-1}{6} - \frac{1-1}{6} = \frac{1}{6} \)

5. \( P(1 \leq X \leq 2) = P(X = 1) + F_X(2) - F_X(1) = 0 + \frac{2-1}{6} - \frac{1-1}{6} = \frac{1}{6} \)

6. \( P(3 < X < 5) = F_X(5) - F_X(3) = \frac{5-1}{6} - \frac{3-1}{6} = \frac{1}{3} \)

Notice that we have used the fact that \( P(3 < X < 5) = P(3 < X \leq 5) \), since \( P(X = 5) = 0 \).

7. \( P(6 < X) = 1 - P(X \leq 6) = 1 - \frac{6-1}{6} = \frac{1}{6} \)
Let \( X \) be a continuous random variable with an exponential distribution. If \( E(X) = 1 \), what is

(8) \( P(X = 2) \)
(9) \( P(X \leq 2) \)
(10) \( P(X > 2) \)
(11) \( P(1 < X \leq 2) \)
(12) \( P(1 \leq X \leq 2) \)
(13) \( P(3 < X < 5) \)
(14) \( P(6 < X) \)

Recall: For an exponential distribution: \( F_X(x) = P(X \leq x) = 1 - \exp\left(-\frac{x}{\alpha}\right) \), where \( \alpha = E(X) \).

**Solutions (8-14):** Using \( \alpha = E(X) = 1 \) in the above formula for the cumulative distribution yields

\[
F_X(x) = P(X \leq x) = 1 - \exp\left(-\frac{x}{\alpha}\right) = 1 - e^{-x},
\]

\[
P(X > x) = 1 - F_X(x) = e^{-x}, \text{ and}
\]

\[
P(c < X \leq d) = F_X(d) - F_X(c) = (1 - e^{-d}) - (1 - e^{-c}) = e^{-c} - e^{-d}.
\]

(8) \( P(X = 2) = 0 \) (because the probability \( P(X = x) = 0 \) for every continuous random variable).

(9) \( P(X \leq 2) = F_X(2) = 1 - e^{-2} \approx .8647 \)

(10) \( P(X > 2) = e^{-2} \approx .1353 \)

(11) \( P(1 < X \leq 2) = F_X(2) - F_X(1) = e^{-1} - e^{-2} \approx .2325 \)

(12) \( P(1 \leq X \leq 2) = P(X = 1) + F_X(2) - F_X(1) = 0 + e^{-1} - e^{-2} \approx .2325 \)

(13) \( P(3 < X < 5) = F_X(5) - F_X(3) = e^{-3} - e^{-5} \approx .0430 \)

Notice that we have used the fact that \( P(3 < X < 5) = P(3 < X \leq 5) \), since \( P(X = 5) = 0 \).

(14) \( P(6 < X) = e^{-6} \approx .0025 \)
15. (uniform distributions: waiting on a friend) Every day Bugs Bunny’s wife leaves her home at 5:00 pm to pick him up from work. With no traffic, she can make it in 20 minutes. With heavy traffic it can take up to 30 minutes. Bugs has been keeping track of Mrs. Bunny’s arrival times, and notices that she is equally likely to arrive at any time between 20 and 30 minutes after 5pm. What is the probability that it takes longer than 25 minutes for her to arrive?

Solution: The arrival time of Mrs. Bunny can be modeled with a uniform distribution. Let $X$ be the arrival time of Mrs. Bunny. Then $X$ is a uniform random variable with $a = 0$ and $b = 10$ (we need not involve the exact time, just the interval over which we are interested). We want $P(X > 5) = \frac{10 - 5}{10 - 0} = \frac{1}{2}$. A second way to do this problem is to let $a = 20$ and $b = 30$. Then we want $P(X > 25) = \frac{30 - 25}{30 - 20} = \frac{1}{2}$. Notice that the answers are the same (as they must be since one is just the translation in time of the other—that is, we just redefine our starting point!).

16. (uniform distributions) A train is scheduled to arrive every hour on the hour. It is typically no more than 5 minutes early or late and is equally likely to arrive anytime over this 10-minute interval. What is the probability that the train is at least 3 minutes early?

Solution: The arrival time of train can be modeled with a uniform distribution. Let $X$ be the arrival time of train. Then $X$ is a uniform random variable with $a = -5$ and $b = 5$, with negative values denoting time before the top of the hour and positive values denoting time after the hour (we need not involve the exact time, just the interval over which we are interested). For the train to be at least 3 minutes early, it must arrive somewhere between 5 minutes and 3 minutes before the top of the hour (i.e. $-5 \leq X \leq -3$ ). We want

$$P(-5 \leq X \leq -3) = P(X \leq -3) = F_X(-3) = \frac{-3 - (-5)}{5 - (-5)} = \frac{2}{10} = \frac{1}{5}.$$ 

17. (uniform distributions) A 100-ft water pipe runs from your toilet to the alley where it connects with the city’s water line. 20 feet of the pipe lies under your house. By law, you are responsible for repairing any leaks in the pipe. To fix a leak under your house is very expensive. Provided that your house is well constructed, a leak is equally likely to occur anywhere along the pipe. If the pipe springs a leak, what is the probability that it is under your house?

Solution: The location of the break in your pipe can be modeled with a uniform distribution. Let $X$ be the location of the leak. Then $X$ is a uniform random variable with $a = 0$ and $b = 100$, with $a = 0$ denoting the starting position under your toilet, and $b = 100$ denoting the end of the pipe that connects with the city’s waterline. We want to calculate the probability of a leak in the first 20 feet of the pipe (the part of the pipe under your house). This is given by

$$P(X \leq 20) = F_X(20) = \frac{20}{100} = \frac{1}{5}.$$
18-20. (uniform distributions) In certain experiments, the error made in determining the density of a substance is a continuous uniform random variable \( X \) with \( b = -a = .15 \). **HINT:** It might help to draw the probability density function \( y = f_X(x) \).

**Solution:** (18-20) Recall: for all continuous random variables \( P(c \leq X \leq d) = F_X(d) - F_X(c) \).

In the case of the uniform distribution we have \( P(X \leq x) = F_X(x) = \frac{x - a}{b - a} \). This means that \( P(c \leq X \leq d) = F_X(d) - F_X(c) = \frac{d - a}{b - a} - \frac{c - a}{b - a} = \frac{d - c}{b - a} \).

We will use this formula throughout this homework set.

18. What is \( P(0 \leq X \leq .01) \)?

**Solution:** Given: \( a = -0.15, \ b = 0.15, \ c = 0, \) and \( d = .01 \). Thus

\[
P(0 \leq X \leq .01) = \frac{.01 - 0}{.15 - (-.15)} = \frac{.01}{.30} = \frac{1}{30}.
\]

19. What is \( P(-.005 \leq X \leq .005) \)?

**Solution:** Given: \( a = -0.15, \ b = 0.15, \ c = -.005, \) and \( d = .005 \). Thus

\[
P(-.005 \leq X \leq .005) = \frac{.005 - (-.005)}{.15 - (-.15)} = \frac{.01}{.30} = \frac{1}{30}.
\]

20. Find the probability that the size of the error will exceed .005.

**Solution:** Since we are only interested in the size of the error, we must use the absolute value of the random variable. We want

\[
P(|X| > .005) = 1 - P(|X| \leq .005) = 1 - P(-.005 \leq X \leq .005) = 1 - \frac{1}{30} = \frac{29}{30}.
\]

(by complementation)

(by definition of absolute value)
21. **(exponential distributions)** Let $T$ be a random variable giving the lifetime (in hours) of the CPU in your computer. The manufacturer claims that $T$ has an exponential distribution with the parameter $\alpha = 10000$ hours. What is the probability that your CPU lasts for at least 5000 hours?

**Solution:** When we work with exponential distributions we need to know the parameter $\alpha$ and the range value $x$. We are given that $\alpha = 10,000$ and $x = 5,000$. We want to determine the probability that the CPU lasts for at least 5000 hours, that is $P(T \geq 5000)$.

For any range value $x$ we have

$$P(X \geq x) = P(X > x) \quad \text{(since } X \text{ is a continuous random variable)}$$

$$= 1 - P(X \leq x) \quad \text{(by complementation)}$$

$$= 1 - F_X(x) \quad \text{(by definition of the cumulative distribution)}$$

$$= \frac{\alpha}{x} \quad \text{(by definition of the exponential function—see class notes)}$$

Substituting our values into this formula yields

$$P(T > 5000) = \exp\left(-\frac{5000}{10,000}\right) = \exp\left(-\frac{1}{2}\right) \approx .6065 .$$

22. **(exponential distributions)** A certain kind of appliance requires repairs an average of once every two years. Assuming that the times between repairs are exponentially distributed, what is the probability that such an appliance will work for at least 3 years without needing repairs?

**Solution:** Let $X$ be the time between repairs. We are given that $\alpha = E(X) = 2$ years, and that $x = 3$ years. We want the probability that $X$ is at least 3 (i.e. $P(X \geq 3)$). Thus

$$P(X \geq 3) = P(X > 3) = \exp\left(-\frac{3}{2}\right) = 0.2231 .$$

23. **(exponential distributions)** A tire company gives a 2-year warranty on its most durable tires. If the tire fails in that time (flats included), then the company will fix or replace the tire. Studies have shown that the tires never fail due to wear and tear over the first two years of their life. We can therefore treat the time until failure of the tire over the two year period as a waiting time problem (waiting for a nail or some other destructive object to mess up the tire). In particular, the mileage (in thousands of miles) that a car owner gets with one of these tires can be modeled as a continuous random variable with an exponential distribution. A study was conducted and it was found that the average mileage until failure for one of these tires is 20,000 miles. Find the probability that one of these tires will last at least 20,000 miles.

**Solution:** We are given that $\alpha = E(X) = 20$ thousand miles, and that $x = 20$ thousand miles. We want the probability that $X$ is at least 20 (i.e. $P(X \geq 20)$). Thus

$$P(X \geq 20) = P(X > 20) = \exp\left(-\frac{20}{20}\right) = \exp(-1) = 0.3679 .$$
Intermediate-Level Problems

24-26. (exponential distributions) A new building has just opened on campus. Unfortunately, the administration has forgotten to budget for replacement light bulbs. The lifetime of a bulb has an exponential distribution with a mean of 2000 hours.

24. If there are 500 bulbs in the building, how long will it be before 100 of them have burned out?

Solution: Let $X = \text{waiting time until the bulbs burn out (out of 100\%)}$. For example, $X = 20$ is the time that 20\% of the bulbs have burn out. We are given that $\alpha = E(X) = 2000$ hours. We want the time $x$ at which 100 bulbs burn out. This is equivalent to the time that it takes for 100 out of the 500 bulbs to burn out, or 1/5 of the total number of bulbs. Thus, we want the time $x$ such that

$$P(X \leq x) = \frac{1}{5}.$$ 

This is equivalent to the question: at what time will $\frac{4}{5}$ of the bulbs still be working (surviving)?

That is, we want $x$ such that $P(X > x) = \frac{4}{5}$.

Using the definition of the exponential distribution gives

$$P(X > x) = \text{Exp}\left(-\frac{x}{\alpha}\right) = \text{Exp}\left(-\frac{x}{2000}\right) = \frac{4}{5}.$$ 

Solving for $x$ yields the desired time:

$$\text{Exp}\left(-\frac{x}{2000}\right) = \frac{4}{5}$$

$$\Rightarrow \ln\left(\text{Exp}\left(-\frac{x}{2000}\right)\right) = \ln\left(\frac{4}{5}\right) \text{ (taking the natural log of both sides)}$$

$$\Rightarrow -\frac{x}{2000} = \ln\left(\frac{4}{5}\right) \text{ (using the property of exponents } \ln(e^a) = a \ln(e) = a)$$

$$\Rightarrow x = -2000 \ln\left(\frac{4}{5}\right) = 2000 \ln\left(\frac{5}{4}\right) = 2000 \ln\left(1 + \frac{1}{4}\right) \approx 2000 \cdot \frac{1}{4} = 500 \text{ hours.}$$
25. If there are 500 bulbs in the building when it opens, how many will be left after 1200 hours?

**Solution:** Let $X$ = waiting time until the bulbs burn out (out of 100%). Since we are given that $\alpha = E(X) = 2000$ hours, we can use the exponential distribution to compute the percentage of bulbs that are working after 1200 hours.

$$P(X > 1200) = \exp \left( - \frac{x}{\alpha} \right) = \exp \left( - \frac{1200}{2000} \right) = \exp \left( - \frac{3}{5} \right) \approx .5488.$$ Thus, we should expect there will be approximately $500(.5488) = 274$ bulbs working after 1200 hours of use.