Mathematics for Business Decisions, Part I

Homework Set 6: Conditional Probability and Independence SOLUTIONS

NOTE: For more practice problems with solutions, see my practice problem sets and my class notes handout.

Elementary-Level Problems

Problems 1-5: Let $E$ and $F$ be two events.

1. **(Conditional Probability)** Suppose that $P(E) = .4$, $P(F) = .5$, and $P(E \cap F) = .2$. Compute $P(E \mid F)$.

   **Solution:** Applying the definition of conditional probability yields
   
   \[
P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{.2}{.5} = \frac{2}{5} = .4.
   \]

2. **(Conditional Probability)** Suppose that $P(E) = .4$, $P(F) = .5$, and $P(E \cap F) = .2$. Compute $P(F \mid E)$.

   **Solution:** Applying the definition of conditional probability yields
   
   \[
P(F \mid E) = \frac{P(E \cap F)}{P(E)} = \frac{.2}{.4} = \frac{1}{2} = .5.
   \]

Notice that $P(E \mid F) \neq P(F \mid E)$. This is generally the case, given two events $E$ and $F$. That means that when we write conditional probabilities, the order of the events is important. We can’t interchangeably write $P(E \mid F)$ when we mean $P(F \mid E)$, and vice versa. This makes real-world sense. Suppose $E$ is the event that a person is a woman, and $F$ is the event that a person is Condoleezza Rice. Then $P(E \mid F) = 1$ (if someone’s Condoleezza Rice, then she’s certainly a woman), but $P(F \mid E)$ is a very small number—out of the 3 billion or so women in the world, only one is Condoleezza Rice. Be careful! order matters!

3. **(Independence)** Suppose that $P(E) = .4$, $P(F) = .5$ and that $E$ and $F$ are independent events. Compute $P(E \cap F)$.

   **Solution:** Applying the definition of independent events
   
   \[
P(E \cap F) = P(E)P(F) = (.4)(.5) = .2.
   \]
4. (Independence) Suppose that $P(E) = .5$, $P(F) = .8$ and that $E$ and $F$ are independent events. Compute $P(E \mid F)$.

Solution: Applying the definition of conditional probability (first equality) together with the definition of independence (the second equality) yields

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E) = .5$$

Recall: A second definition of independence is two events $E$ and $F$ are independent if $P(E \mid F) = P(E)$. Thus a second way to solve this problem is just to write $P(E \mid F) = P(E) = .5$.

5. (Independence) Suppose that $P(E) = .5$, $P(F) = .8$ and that $E$ and $F$ are independent events. Compute $P(F \mid E)$.

Solution: Applying the definition of independent events

$$P(F \mid E) = P(F) = .8.$$ 

Since $F$ is independent of $E$, the probability of $F$ does not change if the event $F$ is conditioned on the event $E$. 
Problems 6-10: Consider the experiment of rolling a fair die twice. All of the 36 outcomes in the sample space $S$ are equally likely. We define the following events. Let $E$ be the event that the first face is odd; let $F$ be the event that the sum of the faces is greater than 8; and let $G$ be the event that the sum of the faces is even.

$$
\Omega = \begin{bmatrix}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{bmatrix}
$$

Solution: The matrix of the sum of the face values, denoted by $S$, is:

$$
S = \begin{bmatrix}
2 & 3 & 4 & 5 & 6 & 7 \\
3 & 4 & 5 & 6 & 7 & 8 \\
4 & 5 & 6 & 7 & 8 & 9 \\
5 & 6 & 7 & 8 & 9 & 10 \\
6 & 7 & 8 & 9 & 10 & 11 \\
7 & 8 & 9 & 10 & 11 & 12
\end{bmatrix}
$$

Hint: Since all of the events are equally likely you may want to determine the following values:

$$\begin{align*}
\#(E) &= 18 \\
\#(F) &= 10 \\
\#(G) &= 18 \\
\#(E \cap F) &= 4 \\
\#(E \cap G) &= 9 \\
\#(F \cap G) &= 4 \\
\#(\Omega) &= 36
\end{align*}$$

We now use the fact that all of the events are equally likely to compute the desired probabilities.


Solution: $P(E | G) = \frac{\#(E \cap G)}{\#(G)} = \frac{9}{18} = \frac{1}{2}$

7. (Conditional Probability) Compute $P(E|F)$.

Solution: $P(E | F) = \frac{\#(E \cap F)}{\#(F)} = \frac{4}{10} = \frac{2}{5}$

8. (Conditional Probability) Compute $P(F|G)$.

Solution: $P(F | G) = \frac{\#(F \cap G)}{\#(G)} = \frac{10}{18} = \frac{5}{9}$
9. **(Independence)** Are the events $E$ and $G$ independent? Recall: Two events are independent if $P(E \cap F) = P(E)P(F)$. An alternative definition is $P(E \mid G) = P(E)$.  
**WARNING:** Do not guess! You must use the definition of independence given above to get any credit!

**Solution:**

$$P(E \mid G) = \frac{\#(E \cap G)}{\#(G)} = \frac{9}{18} = \frac{1}{2} \quad \text{and} \quad P(E) = \frac{\#(E)}{\#(\Omega)} = \frac{18}{36} = \frac{1}{2}.$$  
Since $P(E \mid G) = P(E)$, the events are independent.

10. **(Independence)** Are the events $E$ and $F$ independent? Recall: Two events are independent if $P(E \cap F) = P(E)P(F)$. An alternative definition is $P(E \mid F) = P(E)$.

**WARNING:** Do not guess! You must use the definition of independence given above to get any credit!

**Solution:**

$$P(E \mid F) = \frac{\#(E \cap F)}{\#(F)} = \frac{4}{10} = \frac{2}{5} \quad \text{and} \quad P(E) = \frac{\#(E)}{\#(\Omega)} = \frac{18}{36} = \frac{1}{2}.$$  
Since $P(E \mid F) \neq P(E)$, the events are **not** independent.

11-13. **(Independence)** Consider three identical boxes that we shall refer to as box1, box2, and box3. Each box contains 100 parts, one of which is defective. You randomly choose one part from each box. Start by writing down the events and their associated probabilities.

**Solution:**

**Step 1:** Write down the events.

Let

$$D_i = \text{the event of choosing a defective part from box } i$$

$$D_i^C = \text{the event of choosing a non-defective part from box } i$$

**Step 2:** Write down what you know.

$$P(D_i) = .01 \quad \text{and} \quad P(D_i^C) = 1 - P(D_i) = .99$$

**NOTE:** The first two steps are the same for problems 16-18. The key fact to be aware of in this problem is that all of our selections are independent. Our selection from box1 has no bearing on our selection in box2, or box3.

**11.** What is the probability of selecting 3 non-defective parts?

**Solution:** To choose three non-defective parts you must choose a non-defective part from box1 and a non-defective part from box2 and a non-defective part from box3. This can be expressed more compactly by the set: $D_1^C \cap D_2^C \cap D_3^C$. We want to compute $P(D_1^C \cap D_2^C \cap D_3^C)$. Using the independence assumption:

$$P(D_1^C \cap D_2^C \cap D_3^C) = P(D_1^C)P(D_2^C)P(D_3^C) = (.99)(.99)(.99) = (.99)^3 \approx .9703$$
12. What is the probability of selecting a non-defective part from box 1, a defective part from box 2, and a non-defective part from box 3?

**Solution:** We seek the event \( D_1^C \cap D_2 \cap D_3^C \). We want to compute \( P(D_1^C \cap D_2 \cap D_3^C) \). Using the independence assumption:

\[
P(D_1^C \cap D_2 \cap D_3^C) = P(D_1^C)P(D_2)P(D_3^C) = (.99)(.01)(.99) = (.99)^2(.01) \approx .0098
\]

13. What is the probability of selecting 1 defective part and 2 non-defective parts?

**Solution:** We can select one defective part and 2 non-defective parts in 3 different ways:

\[
D_1 \cap D_2^C \cap D_3^C \quad \text{(the defective part comes from box 1)}
\]

\[
D_1^C \cap D_2 \cap D_3^C \quad \text{(the defective part comes from box 2)}
\]

\[
D_1^C \cap D_2^C \cap D_3 \quad \text{(the defective part comes from box 3)}
\]

We want the probability of selecting \( D_1 \cap D_2^C \cap D_3^C \), or \( D_1^C \cap D_2 \cap D_3^C \), or \( D_1^C \cap D_2^C \cap D_3 \):

\[
P(D_1 \cap D_2^C \cap D_3^C) \cup (D_1^C \cap D_2 \cap D_3^C) \cup (D_1^C \cap D_2^C \cap D_3)
\]

\[
= P(D_1 \cap D_2^C \cap D_3^C) + P(D_1^C \cap D_2 \cap D_3^C) + P(D_1^C \cap D_2^C \cap D_3)
\]

\[
= 3P(D)(P(D^C))^2
\]

\[
\approx 3(.0098) = .0294,
\]

where the event \( D \) is short-hand for \( D_1, D_2, \) or \( D_3 \).


14. Find the probability that the second flip lands on heads.

**Solution:** Let \( H_i \) be the event of heads on the \( i^{th} \) flip for \( i = 1, 2 \). Since the outcomes from the two flips are independent, it follows that \( P(H_2) = .5 \).

15. Find the probability that the second toss is a head, given that the first toss was heads.

**Solution:** Let \( H_i \) be the event of heads on the \( i^{th} \) flip for \( i = 1, 2 \). Since the outcomes from the two flips are independent, we have \( P(H_2 \mid H_1) = P(H_2) = .5 \). This is the same answer as the previous problem. We should have expected this because the outcomes of the tosses are independent.
Intermediate-Level Problems

16. (Conditional Probability) Let $E$ and $F$ be any two events. Is it always true that $P(E \ | \ F) = P(F \ | \ E)$? To receive any credit you must justify your answer correctly. What we are asking here is: Given two events, can we just switch or reverse the order of the event that we are conditioning on without changing the probability?

**HINT:** Use the definition of conditional probability $P(E \ | \ F) = \frac{P(E \cap F)}{P(F)}$ on each expression.

**Solution:** Start by setting the two equal:

$$P(E \ | \ F) = P(F \ | \ E)$$

Using the definition of conditional probability this statement becomes

$$\frac{P(E \cap F)}{P(F)} = P(E \ | \ F) = P(F \ | \ E) = \frac{P(E \cap F)}{P(E)}.$$

If $P(E \cap F) \neq 0$, then we can divide by $P(E \cap F)$ to arrive at

$$\frac{1}{P(F)} = \frac{1}{P(E)}.$$

Upon inverting the expression we arrive at $P(E) = P(F)$.

If $P(E \cap F) = 0$, then the events are mutually exclusive and it follows that both of the terms $P(E \ | \ F)$ and $P(F \ | \ E)$ vanish. For example, if $F$ occurs, then $E$ cannot occur (the events are mutually exclusive), and so the probability of $E$ given $F$ is zero.

Moreover, if this expression were always true, then why would we need Bayes’ theorem to reverse the order in which we condition the probabilities?
17. (Conditional Probability) In a certain region of the country, atmospheric conditions are classified into a finite number of categories, and weather records involving the events of rain and gale-force winds are kept for each of these categories. Weather records indicate that under category 1 atmospheric condition there is a 40% chance of rain, a 30% chance of gale-force winds, and a 50% chance that it either rains or has gale-force winds. Suppose you wake up on a day with category 1 conditions and find that it is raining. What is the probability of gale-force winds? (Assume that you know the all of the categories and can classify them correctly.)

Solution: Step 1: Write down the events.

Let
\[
\begin{align*}
R &= \text{the event of rain} \\
W &= \text{the event of gale-force winds}
\end{align*}
\]

Step 2: Write down what you know.

\[P(R) = .4, \quad P(W) = .3, \quad \text{and} \quad P(R \cup W) = .5.\]

Step 3: Write down what you want.

We are told that it is raining. Thus the event \(R\) has occurred. This means that we want to know the probability of gale-force winds given that it is raining. In symbols, we want \(P(W \mid R)\).

Step 4: Solve

Using the definition of conditional probability and \(P(R) = .4\) leads to the following equation:

\[
P(W \mid R) = \frac{P(R \cap W)}{P(R)} = \frac{P(R \cap W)}{.4}. \tag{Equation 1}
\]

We cannot evaluate the expression on the right-hand side because we don’t know \(P(R \cap W)\). However, we can use the formula \(P(R \cup W) = P(R) + P(W) - P(R \cap W)\) to solve for \(P(R \cap W)\). Substituting the values found in step 2 into the formula yields

\[.5 = .4 + .3 - P(R \cap W).\]

Solving for \(P(R \cap W)\) in the equation gives us \(P(R \cap W) = .2\). For the last step, we must substitute this value in to equation 1 to find the desired result.

\[
P(W \mid R) = \frac{P(R \cap W)}{.4} = \frac{.2}{.4} = \frac{1}{2}
\]
18. (Conditional Probability) In 95% of all manned lunar flights, a midcourse trajectory correction is required. This is done by sending a fire signal from ground control to ignite small correction thrusters. For technical reasons, this fire signal is sometimes not executed by the thrusters. Tests show that the probability is .0001 that a fire signal will not be executed when it is required. If the correction is required and not executed, then the rocket will be not be able to escape the sun’s gravitational field and will be sucked into the sun, which will make the astronauts unhappy. What is the probability that this will happen?

Solution: Step 1: Write down the events.

Let \( E = \) the event correction is not executed
\( R = \) the event correction is required

Step 2: Write down what you know.

\( P(R) = .95, \quad P(\text{correction is not executed} \mid \text{correction is required}) = P(E \mid R) = .0001. \)

Step 3: Write down what you want.

We want to know the probability that a correction is required \textit{and} that the correction is not executed.

Step 4: Solve

To find \( P(R \cap E) \), we cannot use the formula \( P(R \cup E) = P(R) + P(E) - P(R \cap E) \), because we do not know \( P(E) \) and \( P(R \cup E) \). This is a conditional probability problem so we must use the definition of conditional probability. Because of the given data, we condition on \( R \). This leads to the following solution:

\[
P(E \cap R) = P(R)P(E \mid R) = (.95)(.0001) = .000095.
\]
19. (Conditional Probability) A company’s yearly report showed that 70% of its employees missed no work, 20% took one sick day, 9% took two sick days, and 1% took three or more sick days from work. What is the probability that an employee who missed work only took one sick day?

**Solution:** Step 1: Write down the events.

Define the following events:

- $M$ = the event that the employee missed at least one day of work over the year
- $S_0$ = the event that the employee took no sick days (missed no work: $M^C$)
- $S_1$ = the event that the employee took one sick day
- $S_2$ = the event that the employee took two sick days
- $S_3$ = the event that the employee took three or more sick days

**NOTE 1:** It is not necessary to define all of these events. I wrote them out for bookkeeping purposes only.

**NOTE 2:** $M = S_1 \cup S_2 \cup S_3$ and $M^C = S_0$.

Step 2: Write down what you know.

$$P(M) = 1 - P(M^C) = 1 - P(S_0) = 1 - .7 = .3, \quad P(S_1) = .2, \quad P(S_2) = .09, \quad P(S_3) = .01.$$  

Step 3: Write down what you want.

We want to know the probability that a person used only one sick day given that they missed at least one day of work. In our mathematical notation this becomes

$$P(\text{missed exactly one day of work} | \text{missed at least one day of work}) = P(S_1 | M).$$

Step 4: Solve

$$P(S_1 | M) = \frac{P(S_1 \cap M)}{P(M)} = \frac{P(S_1)}{P(M)} = \frac{.2}{.3} = \frac{2}{3}$$

**NOTE:** Since the event $S_1$ is a subset of the event $M$ (e.g. $S_1 \subseteq M$) we have that $S_1 \cap M = S_1$. 


20. (Independence) A card is drawn from a standard deck of 52 cards. The card is displayed on the table and is not placed back in the deck. This is known as choosing a card without replacement. A second card is then drawn from the deck. Determine the probability that the second card is a spade, given that the first card is a spade.

*NOTE:* There are 13 spades in a standard deck of 52 cards.

**Solution:**

*Step 1:* Write down the events.

Define the following events: Let

- $S_1$ be the event that the first card chosen is a spade
- $S_2$ be the event that the second card chosen is a spade

*Step 2:* Write down what you know.

Provided the deck is well shuffled between each pick, each card is equally likely to be picked. Thus

$$P(S_1) = \frac{13}{52}.$$  

If a spade is chosen on the first pick and that the card was not replaced (we are choosing the second card without replacement), then on the second pick there are only 51 cards to choose from and only 12 of those cards are spades.

*Step 3:* Write down what you want.

We want the probability of picking a spade on the second pick, given that we have chosen a spade on the first pick. Thus

$$P(S_2 \mid S_1) = \frac{12}{51}.$$