Solutions to Selected Problems
from the text:
Mathematics for Business Decisions

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Contents

1 Project 1 Material .......................... 2
  1.1 Summation .................................. 2
  1.2 Basic Probability ........................... 3
  1.3 Expected Value .............................. 8
  1.4 Conditional probability and independence .......... 11
  1.5 Partitions .................................. 13
  1.6 Bayes’ theorem ............................. 16

2 Material for Project 2 .................. 19
  2.1 Compound Interest ........................... 19
  2.2 Finite Probability Distributions ................. 31
  2.3 Continuous Probability Distribution ............. 33
1 Project 1 Material

1.1 Summation

In problems 1-8 in the book *Mathematics for Business Decisions* by Thompson and Lamoureux, you are asked to evaluate the given sums. (In some cases, you are to do them by hand; in other cases, using Excel). Below are the solutions to these problems.

**Exercise 1 (Summation):**

\[
\sum_{i=3}^{6} (i - 3)^2 = (3 - 3)^2 + (4 - 3)^2 + (5 - 3)^2 + (6 - 3)^2 \\
= 0^2 + 1^2 + 2^2 + 3^2 = 14
\]

**Exercise 2 (Summation):**

\[
\sum_{i=1}^{6} (i^2 - 1) \\
= (1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) + (5^2 - 1) + (6^2 - 1) \\
= 0 + 3 + 8 + 15 + 24 + 35 = 85
\]

**Exercise 3 (Summation):**

\[
\sum_{k=1}^{5} \left(\frac{k}{2}\right)^2 = \frac{1}{4} \sum_{k=1}^{5} k^2 = \frac{1}{4} \left(1^2 + 2^2 + 3^2 + 4^2 + 5^2\right) = \frac{55}{4}
\]

**Exercise 4 (Summation):**

\[
\sum_{j=0}^{100} j = \frac{100(101)}{2} = 50.5
\]

(Use the Gauss summation formula \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\) from the notes.)

**Exercise 5 (Summation):** Write 0.5 as a fraction 1/2. Then

\[
\sum_{k=1}^{30} \left(\frac{1}{2}\right)^k = \frac{1}{2} \sum_{j=0}^{29} \left(\frac{1}{2}\right)^j = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^{30}}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^{30}
\]

(We let \(j = k - 1\), and used the formula for a geometric series.)

**Exercise 6 (Summation):**

\[
\sum_{n=1}^{500} \frac{1}{n} \approx \ln(500) + .577, \text{ where we have used Euler’s approximation. You should verify this using Excel. See also problem-set2-summation on my website.}
\]

**Exercise 7 (Summation):**

\[
\sum_{j=1}^{5} j^2 + 2 \sum_{j=1}^{5} j = \sum_{j=1}^{5} (j^2 + 2j) = \sum_{i=1}^{5} (i^2 + 2i)
\]

(We could add the first two sums because they both ran from \(j = 1\) to \(j = 5\). In the last expression, we substituted \(i = j\): the precise name of the index of summation doesn’t matter.)
Exercise 8 (Summation): \[ \sum_{j=0}^{3} (2j+1) = 2 \sum_{k=1}^{4} (2(k-1) + 1) = \sum_{k=1}^{4} (2k-1) \]

(Here we let \( k = j + 1 \). If \( 0 \leq j \leq 3 \), then \( 1 \leq j + 1 = k \leq 4 \).)

**NOTE:** I did not use Excel to solve any of these problems. However, you should use Excel to verify the answers I’ve given here. You can find solutions (using Excel) to similar problems on my website: http://dtc.pima.edu/hacker/ under business math, problem-set2-summation-sols.xls.

### 1.2 Basic Probability

**Problem 5 (Basic Probability):** For part (i), we want the probability of inspection on a given day. Here

\[ \Omega = \{ \text{Sun, Mon, Tues, Wed, Thurs, Fri, Sat} \} \]

The probability that an inspection will happen on a particular day is equally likely for all the days. We’ll denote that probability by \( p \). Then

\[ p = P(\{\text{Sun}\}) = P(\{\text{Mon}\}) = \cdots P(\{\text{Sat}\}) \]

Then

\[ 1 = P(\Omega) = P(\{\text{Sun, Mon,} \ldots, \text{Sat}\}) \]
\[ = P(\{\text{Sun}\} \cup \{\text{Mon}\} \cup \cdots \cup \{\text{Sat}\}) \]
\[ = P(\{\text{Sun}\}) + \cdots + P(\{\text{Sat}\}) \]
\[ \quad \text{(since the subsets } \{\text{Sun}\}, \ldots, \{\text{Sat}\} \text{ are disjoint)} \]
\[ = p + p + \cdots + p \quad (7 \text{ times}) \]
\[ = 7p \]

When we solve for \( p \), we get

\[ p = \frac{1}{7} \]

For part (ii), let \( E \) be the event that the inspection is on a weekday. Then

\[ E = \{ \text{Mon, Tues,} \ldots, \text{Fri} \} = \{ \text{Mon} \} \cup \cdots \cup \{ \text{Fri} \} \]

\[ P(E) = P(\{\text{Mon}\} \cup \cdots \cup \{\text{Fri}\}) \]
\[ = P(\{\text{Mon}\}) + \cdots + P(\{\text{Fri}\}) \]
\[ = p + p + p + p + p = 5p \]
\[ = 5 \left( \frac{1}{7} \right) = \frac{5}{7} \]
Problem 6 (Basic Probability): We’ll abbreviate: \( J=\text{Jan}, \ F=\text{Feb}, \ M=\text{Mar} \). Then \( \Omega = \{J, F, M\} \). We need to list all the events of \( \Omega \), i.e. all the subsets of \( \Omega \).

<table>
<thead>
<tr>
<th>Event</th>
<th>Description of event</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset = {} )</td>
<td>No job offer in January, February, or March</td>
</tr>
<tr>
<td>( {J} )</td>
<td>Job offer in January</td>
</tr>
<tr>
<td>( {F} )</td>
<td>Job offer in February</td>
</tr>
<tr>
<td>( {M} )</td>
<td>Job offer in March</td>
</tr>
<tr>
<td>( {J, F} )</td>
<td>Job offer in January or February</td>
</tr>
<tr>
<td>( {J, M} )</td>
<td>Job offer in January or March</td>
</tr>
<tr>
<td>( {F, M} )</td>
<td>Job offer in February or March</td>
</tr>
<tr>
<td>( {J, F, M} )</td>
<td>Job offer in January, February, or March</td>
</tr>
</tbody>
</table>

We have listed all the elements in the power set of \( \Omega \). \( \#(\Omega) = 3 \), so the number of elements in the power set is

\[
\#(\mathcal{P}(\Omega)) = 2^{\#(\Omega)} = 2^3 = 8.
\]

Problem 7 (Basic Probability): We’ll use the notation from Example 3 on page 26, with the events:

- \( F = \text{business fails during the first year} \)
- \( V = \text{business is owner’s first venture} \)

We want the probability of the event \((F^c \text{ and } V^c)\): the business does not fail during its first year, and it is not its owner’s first venture.

Recall that when we use the word “and” in connection with events, we take the intersection \( \cap \). Thus we want

\[
\mathcal{P}(F^c \cap V^c) = \mathcal{P}((F \cup V)^c) \quad \text{(by DeMorgan’s law)}
\]

\[
= 1 - \mathcal{P}(F \cup V) \quad \text{(by fundamental property 5)}
\]

\[
= 1 - 0.71 \quad \text{(from example 3, \( \mathcal{P}(F \cup V) = 0.71 \))}
\]

\[
= 0.29
\]

Problem 8 (Basic Probability): From Example 3 on page 26, we define the following events:

**Step 1.** Let \( F \) be the event that the business fails in its first year. Then \( F^c \) is the event that the business doesn’t fail in its first year.

Let \( V \) be the event that this is the business owner’s first venture. Then \( V^c \) is the event that it is not his first venture.

**Step 2.** Given:

\[
\mathcal{P}(F) = 0.69 \\
\mathcal{P}(V) = 0.21 \\
\mathcal{P}(F \cap V) = 0.19 \\
\mathcal{P}(F \cup V) = 0.71
\]
**Step 3.** We want the probability of the event:
the business is **not** the owner’s first venture, **and** that it fails during the first year
→ **not** \( V \) **and** \( F \) → \( V^c \cap F \).

\[
P(V^c \cap F) = P(F - V) \\
= P(F) - P(F \cap V) \\
= 0.69 - 0.19 \\
= 1/2.
\]

Figure 1:

**Problem 9 (Basic Probability):**

**Step 1.** Define the events:

\( R = \text{get raise} \quad R^c = \text{don’t get raise} \)

\( O = \text{get own office} \quad O^c = \text{don’t get own office} \)

**Step 2.** Write down what you know. Given:

\[
\begin{align*}
P(R) &= 0.68 \\
P(R^c) &= 1 - P(R) = 1 - 0.68 = 0.32 \\
P(O) &= 0.41 \\
P(O^c) &= 1 - P(O) = 1 - 0.41 = 0.59 \\
P(R \cup O) &= 0.85
\end{align*}
\]

The last is the probability of getting a raise **or** one’s own office. “Or” corresponds to the union of events.

**Step 3.** Write down what you want. We want the probability of getting a raise **and**
one’s own office. “And” corresponds to the intersection of events.

\[
P(R \cap O) = P(R) + P(O) - P(R \cup O) \\
= 0.68 + 0.41 - 0.85 \\
= 0.24
\]

**Problem 10 (Basic Probability):**
The first two steps are the same as in Exercise 9. Now, we want the probability that you
get neither a raise **nor** your own office.

Neither \( R \) nor \( O \) = \( R^c \) and \( O^c \) = \( R^c \cap O^c \)

We want: \( P(R^c \cap O^c) \)

\[
\begin{align*}
&= P((R \cup O)^c) \quad \text{(by DeMorgan’s laws)} \\
&= 1 - P(R \cup O) \\
&= 1 - 0.85 \\
&= 0.15
\end{align*}
\]
**Problem 11 (Basic Probability):** Define the following events:

- $B =$ Job at Big
- $H =$ Job at Huge
- $B^c =$ No job at Big
- $H^c =$ No job at Huge
- $B \cup H =$ Job at Big or Huge
- $B \cap H =$ Job at Big and Huge

$P(B) = 0.3, P(H) = 0.5, P(B \cap H) = 0.1$

We want the probability of getting exactly one of the jobs. The set $B \cup H$ is the event of getting at least one of the jobs. It can be broken down into three parts:

- $B - H =$ Job at Big, no job at Huge
- $H - B =$ No job at Big, job at Huge
- $H \cap B =$ Job at Big and job at Huge.

We need to exclude the third possibility, so we subtract out the intersection $B \cap H$. That leaves us with two moon-shaped regions: $B - H$ and $H - B$.

\[
P(\text{exactly one job}) = P(B \cup H) - P(B \cap H) \\
= P(B) + P(H) - 2P(B \cap H) \\
= 0.3 + 0.5 - 2(0.1) = 0.6
\]

**Problem 12 (Basic Probability):** The event “neither job” means (not $B$) and (not $H$) both occur. In other words, $B^c$ and $H^c$ occur. When we see “and”, we think of the intersection: $B^c \cap H^c$

\[
P(B^c \cap H^c) = P((B \cup H)^c) \quad \text{(by DeMorgan’s laws)} \\
= 1 - P(B \cup H) \\
= 1 - 0.7 \quad \text{(from example 3)} \\
= 0.3
\]

**Problem 13 (Basic Probability):**

Define the events:

- $R =$ rain
- $R^c =$ no rain
- $H =$ high winds
- $H^c =$ no high winds

Given:

- $P(R) = 0.3$
- $P(H) = 0.4$
- $P(R \cup H) = 0.5$

We want the probability that there is neither rain nor high winds.

\[
P(R^c \cap H^c) = P((R \cup H)^c) \\
= 1 - P(R \cup H) \\
= 1 - 0.5 \\
= 0.5
\]
Problem 14 (Basic Probability): We write the event “R and not H” as

\[ R \cap H^c = R - H \]

We want \( P(R \cap H^c) \). To get it, we need an expression involving \( R - H \). We can get one by dividing the event \( R \) into two disjoint sets: \( R \cap H^c = R - H \), and \( R \cap H \). These two sets form a partition for the set \( R \). Thus

\[
P(R) = P((R \cap H^c) \cup (R \cap H)) \\
= P(R \cap H^c) + P(R \cap H) \\
\text{(since the union of the two sets is } R) \\
\text{(since the two sets are disjoint)}
\]

Rearranging this gives us

\[ P(R \cap H^c) = P(R) - P(R \cap H) \quad (1.1) \]

To compute \( P(R \cap H) \), use the equation

\[ P(R \cup H) = P(R) + P(H) - P(R \cap H) \]

Substituting the information from Problem 13 yields

\[ 0.5 = 0.3 + 0.4 - P(R \cap H) \]

Solving this equation get us

\[ P(R \cap H) = 0.7 - 0.5 = 0.2 \]

Substituting this into equation (1.1) produces

\[ P(R \cap H^c) = 0.3 - 0.2 = 0.1 \]

Problem 15 (Basic Probability): Here we want the probability of event \( H \) and not event \( R \): \( P(H \cap R^c) = P(H - R) \). This is similar to Problem 14, with the roles of \( H \) and \( R \) reversed. Using the same reasoning that we used in Problem 14, we get

\[ P(H \cap R^c) = P(H) - P(H \cap R) = 0.4 - 0.2 = 0.2 \]
1.3 Expected Value

Problem 4 (Expected value): Your report is evaluated in each of three categories. For each category, you can receive one of three grades: G for “good,” F for “fair,” or P for “poor.” Let $X$ equal the number of G’s your report gets. Suppose that all outcomes are equally likely: that is, in each category, you’re equally likely to get G, F, or P.

Compute (i) $\mathcal{P}(X = 0)$ and (ii) $\mathcal{P}(X = 2)$.

We’ll start by writing down the sample space. In each category, there are 3 possible grades. There are 3 categories in total. The total number of possible evaluations is

\[(3 \text{ choices for Cat. 1}) \times (3 \text{ choices for Cat. 2}) \times (3 \text{ choices for Cat. 3}) = 3^3 = 27.\]

We should expect to find 27 elements in the sample space:

\[S = \{GGG, GGF, GGP, GFG, GFF, GFP, GPG, GPF, GPP, FGG, FGF, FGP, FFG, FFF, FFP, FPG, FPF, FPP, PGG, PGF, PGP, PFG, PFF, PFP, PPG, PPF, PFP, PPP\}\]

(i) The event \{X = 0\} corresponds to the set of outcomes in S that include no G’s. There are eight such outcomes; so

\[\mathcal{P}(X = 0) = \frac{\# \text{ of outcomes with no G’s}}{\text{total # of outcomes}} = \frac{8}{27}.\]

(ii) For $\mathcal{P}(X = 2)$, count the outcomes with exactly two G’s. There are 6; so

\[\mathcal{P}(X = 2) = \frac{\# \text{ of outcomes with exactly 2 G’s}}{\text{total # of outcomes}} = \frac{6}{27}.\]

Problem 6 (Expected value):

The range of $X$ is \{0, 1, 2, 3\}. $\mathcal{P}(X > 1) = \mathcal{P}(X = 2) + \mathcal{P}(X = 3)$

Since each outcome is equally likely, we have

\[\mathcal{P}(X = 2) = \frac{6}{27} \quad \text{(see my solutions to Exercise 4)}\]
\[\mathcal{P}(X = 3) = \frac{1}{27}\]

Notice that

\[\{X = 2\} = \{GGF, GFG, FGG, GGP, GPG, PGG\}\]
\[\{X = 3\} = \{GGG\}\]
\[\#(\Omega) = 27\]

Thus

\[\mathcal{P}(X > 1) = \mathcal{P}(X = 2) + \mathcal{P}(X = 3) = \frac{6}{27} + \frac{1}{27} = \frac{7}{27}.\]
Problem 10 (Expected value):
Let $\tilde{X}$ be the random variable that gives net profit to the state on a single ticket. Then
\[
\text{range of } \tilde{X} = \text{range of } X \quad \text{(from example 6)}
\]
\[
= \{2 - 10^6, 2 - 10^2, 2 - 10, 2\}
\]
\[
\mathcal{P}(\tilde{X} = 2 - 10^6) = \frac{4}{10^7} = \frac{5 - 1}{10^7} = \mathcal{P}(X = 2 - 10^6) - \frac{1}{10^7}
\]
\[
\mathcal{P}(\tilde{X} = 2 - 10^2) = \mathcal{P}(X = 2 - 10^2) = \frac{8}{10^3}
\]
\[
\mathcal{P}(\tilde{X} = 2 - 10) = \mathcal{P}(X = 2 - 10) = \frac{1}{10^2}
\]
\[
\mathcal{P}(\tilde{X} = 2) = 1 - \mathcal{P}(\tilde{X} = 2 - 10^6) - \mathcal{P}(\tilde{X} = 2 - 10^2) - \mathcal{P}(\tilde{X} = 2 - 10)
\]
\[
= 1 - \mathcal{P}(X = 2 - 10^6) + \frac{1}{10^7} - \mathcal{P}(X = 2 - 10^2) - \mathcal{P}(X = 2 - 10)
\]
\[
= \mathcal{P}(X = 2) + \frac{1}{10^7}
\]

We now have the probabilities of $\tilde{X}$ in terms of the probabilities of $X$.
\[
E[\tilde{X}] = (2 - 10^6)\mathcal{P}(\tilde{X} = 2 - 10^6) + (2 - 10^2)\mathcal{P}(\tilde{X} = 2 - 10^2)
\]
\[
+ (2 - 10)\mathcal{P}(\tilde{X} = 2 - 10) + 2\mathcal{P}(\tilde{X} = 2)
\]
\[
= (2 - 10^6) \left[ \mathcal{P}(X = 2 - 10^6) - \frac{1}{10^7} \right] + (2 - 10^2)\mathcal{P}(X = 2 - 10^2)
\]
\[
+ (2 - 10)\mathcal{P}(X = 2 - 10) + 2 \left[ \mathcal{P}(X = 2) + \frac{1}{10^7} \right] \quad \text{(substitute for } X)\)
\]
\[
= E[X] - \frac{1}{10^7} (2 - 10^6) + \frac{2}{10^7}
\]
\[
= E[X] + \frac{10^6 - 2 + 2}{10^7}
\]
\[
= E[X] + \frac{1}{10}
\]
\[
= 0.70
\]

Problem 11 (Expected value):
Let $T$ be the value of the top prize. All other prizes are the same, and their probabilities are the same. Let $X$ be the random variable representing the state’s net profit, as in example 6.

If a $2 ticket wins $T$ dollars, then the state takes in $2 - T$ dollars (a loss). If a $2 ticket wins $100 or $10, then $X = 2 - 10^2$ or $2 - 10$ respectively. A ticket that wins $0 leaves the state with a profit of $X = 2$. Thus the range of $X$ is
\[
\{2 - T, 2 - 10^2, 2 - 10, 2\}\]
and the probabilities are
\[
\mathcal{P}(X = 2 - T) = \frac{5}{10^7},
\]
\[
\mathcal{P}(X = 2 - 10^2) = \frac{8}{10^6},
\]
\[
\mathcal{P}(X = 2 - 10) = \frac{1}{10^2},
\]
\[
\mathcal{P}(X = 2) = 1 - \frac{5}{10^7} - \frac{8}{10^6} - \frac{1}{10^2} = 0.9819995
\]
We demand that \(E[X] = \frac{1}{2}\) (50 cents per ticket). The formula for expected value is:
\[
E[X] = (2 - T)\mathcal{P}(X = 2 - T) + (2 - 10^2)\mathcal{P}(X = 2 - 10^2) + (2 - 10)\mathcal{P}(X = 2 - 10) + 2\mathcal{P}(X = 2)
\]
Substituting the known values, we get
\[
\frac{1}{2} = (2 - T) \frac{5}{10^7} + (2 - 10^2) \frac{8}{10^6} + (2 - 10) \frac{1}{10^2} + 2 \left( 1 - \frac{5}{10^7} - \frac{8}{10^6} - \frac{1}{10^2} \right)
\]
Multiply by \(2 \cdot 10^6\) and obtain
\[
10^6 = (2 - T) \frac{5}{10^7} + (2 - 10^2) \cdot 16 \cdot 10^3 + (2 - 10) \cdot 2 \cdot 10^4 + 4 \cdot 10^6 \left( 1 - \frac{5}{10^7} - \frac{8}{10^6} - \frac{1}{10^2} \right)
\]
Add \(T - 10^6\) to both sides of the equation:
\[
T = (2 - 10^6) + (2 - 10^2) \cdot 16 \cdot 10^3 + (2 - 10) \cdot 2 \cdot 10^4 + 4 \cdot 10^6 \left( 1 - \frac{5}{10^7} - \frac{8}{10^6} - \frac{1}{10^2} \right)
\]
Simplifying yields
\[
T = 2 - 10^6 + 32 \cdot 10^3 - 16 \cdot 10^5 + 4 \cdot 10^4 - 2 \cdot 10^5 + 4 \cdot 10^6 - 2 - 32 \cdot 10^3 - 4 \cdot 10^4
\]
After much cancellation of terms, we get
\[
T = 3 \cdot 10^6 - 18 \cdot 10^5
= (30 - 18)10^5
= 12 \cdot 10^5
= 1.2 \cdot 10^6
= 1,200,000
\]

**Problem 13 (Expected value):**

Let \(X_A\) = return on investment in project A (profit)
Let \(X_B\) = return on investment in project B
Range of \(X_A\) = \{-26000, 0, 68000\}
Range of \(X_B\) = \{-71000, 0, 143000\}
\[ P(X_A = -26000) = 0.3 \quad P(X_B = -71000) = 0.2 \]
\[ P(X_A = 0) = 0.5 \quad P(X_B = 0) = 0.65 \]
\[ P(X_A = 68000) = 0.2 \quad P(X_B = 143000) = 0.15 \]

Compute the expected returns:

\[ E[X_A] = -26000 \cdot P(X_A = -26000) + 0 \cdot P(X_A = 0) + 68000 \cdot P(X_A = 68000) \]
\[ = -26000(0.3) + 68000(0.2) = 5800 \]
\[ E[X_B] = -71000(0.2) + 143000(0.15) = 7250 \]

Since the expected return from project B is greater than the expected return from project A, choose project B.

**Problem 14 (Expected value):**

Constraint: we only have $300 to lose, and we must have 1000 certificates. Let \( n \) be the number of $100 certificates in the box; let \( C \) be a random variable giving the value of the certificate \( (C = 1 \text{ or } 100) \).

\[ E(C) = 1 \cdot P(C = 1) + 100 \cdot P(C = 100) \]
\[ = 1 \cdot \frac{\text{# of $1 bills in box}}{\text{total # of certificates}} + 100 \cdot \frac{\text{# of $100 bills in box}}{\text{total # of certificates}} \]
\[ = 1 \cdot \frac{1000 - n}{1000} + 100 \cdot \frac{n}{1000} \]
\[ = 1 + \frac{99n}{1000}. \]

Given: Expected cost = $100 \( E(C) = $(100 + \frac{99}{10}n)$. 
Set: Expected cost = $300. Then

\[ 300 = 100 + \frac{99}{10}n \]
\[ \frac{200 \cdot 10}{99} = n \]
\[ n \approx \frac{2000}{100} = 20. \]

Of course, to make sure that we didn’t exceed $300, we’d only put in two $100 bills!

### 1.4 Conditional probability and independence

**Problem 1 (Conditional probability and Independence):**

**Step 1:** Define events: let
\( R \) = event that it rains \( R^c = \) not rain
\( W \) = event that it is windy \( W^c = \) not windy

**Step 2:** Write down the given information. From example 1,
\[
\begin{align*}
P(R \cap W) &= 0.24 & P(R) &= 0.3 \\
P(R \cup W) &= 0.46 & P(W) &= 0.4
\end{align*}
\]

**Step 3:** Write down what you want to compute. We want the probability that it is windy, given that it is raining:
\[
P(W \mid R) = \frac{P(W \cap R)}{P(R)} = \frac{0.24}{0.30} = \frac{24}{30} = \frac{4}{5}
\]

**Problem 8 (Conditional probability and Independence):**

**Step 1:** Define the events. Let
\( D_i \) = event of choosing a defective part in the \( i \)th draw
\( D_i^c \) = choosing a non-defective part in that draw

**Step 2:** Write down the given information.
\[
\begin{align*}
P(D_i) &= 0.05 & P(D_i^c) &= 1 - P(D_i) = 0.95
\end{align*}
\]

Since we replace the part we drew before we draw again, each selection is independent; and \( P(D_i) \) is the same for every value of \( i \). To make this easier to read, we’ll replace \( P(D_i) \) and \( P(D_i^c) \) with \( P(D) \) and \( P(D^c) \) when the \( i \) is unnecessary.

**Step 3:** Write down what you are trying to find. We want the probability of selecting a non-defective part on the first, second, and third draws.
\[
P(D_1^c \cap D_2^c \cap D_3^c) = P(D_1^c)P(D_2^c)P(D_3^c) = (0.95)^3
\]

**Problem 9 (Conditional probability and Independence):**

**Step 1.** Define the events:

Let \( D_i = \) Choose a defective part in the \( i \)th draw
\( D_i^c = \) Choose a non-defective part in the \( i \)th draw

**Step 2.** Write down the given information:
\[
\begin{align*}
P(D_i) &= 0.05 & P(D_i^c) &= 1 - 0.05 = 0.95
\end{align*}
\]

**Step 3.** Write down what you are trying to solve for.

(i) The event of selecting a defective part, then a non-defective part, then another non-defective part is \( D \cap D^c \cap D^c \). By independence,
\[
P(D_1 \cap D_2^c \cap D_3^c) = P(D_1)P(D_2^c)P(D_3^c) = (0.05)(0.95)^2
\]

(ii) Let \( E \) be the event of getting exactly one defective part. Then \( E \) can be written as the union of three mutually exclusive events: getting the single defective part on the first selection; getting it on the second selection; and getting it on the third selection.
\[
E = \{D \cap D^c \cap D^c\} \cup \{D^c \cap D \cap D^c\} \cup \{D^c \cap D^c \cap D\}
\]
\[ \mathbb{P}(E) = \mathbb{P}\left(\{D \cap D^c \cap D^c\} \cup \{D^c \cap D \cap D^c\} \cup \{D^c \cap D^c \cap D\}\right) \]
\[ = \mathbb{P}(D \cap D^c \cap D^c) + \mathbb{P}(D^c \cap D \cap D^c) + \mathbb{P}(D^c \cap D^c \cap D) \]
(We can do this because the events are disjoint)
\[ = \mathbb{P}(D)\mathbb{P}(D^c)\mathbb{P}(D^c) + \mathbb{P}(D^c)\mathbb{P}(D)\mathbb{P}(D^c) + \mathbb{P}(D^c)\mathbb{P}(D^c)\mathbb{P}(D) \]
(because the three selections are independent)
\[ = 3\mathbb{P}(D)(\mathbb{P}(D^c))^2 \]
\[ = 3(0.05)(0.95)^2 = 0.135 \]

1.5 Partitions

Problem 4 (Bayes’ Theorem): 
Step 1: Define the events. Let

\[ T = \text{teens} \quad A = \text{adults}\]

Then if \( \Omega \) is the set of all people,

\[ \Omega = T \cup A \quad \text{and} \quad T \cap A = \emptyset \]

Thus \( T \) and \( A \) partition the population \( \Omega \). Next, let

\[ L = \text{listeners} \quad L^c = \text{non-listeners} \]

Step 2: Write down the given information.

\[ \mathbb{P}(T) = 0.29 \quad \text{(probability of randomly selecting a teen)} \]
\[ \mathbb{P}(A) = 0.71 \quad \text{(probability of randomly selecting an adult)} \]
\[ \mathbb{P}(L \mid T) = 0.21 \Rightarrow \mathbb{P}(L^c \mid T) = 1 - \mathbb{P}(L \mid T) = 0.79 \]
\[ \mathbb{P}(L \mid A) = 0.14 \Rightarrow \mathbb{P}(L^c \mid A) = 1 - \mathbb{P}(L \mid A) = 0.86 \]

How did we get the last two lines? If 14% of adults are listeners, then the probability that a random adult will be a listener is 14%. In other words, given that we select an adult, the probability that the person will be a listener is 14%. We can make a similar argument about teens.

(i) Here is a tree diagram illustrating the situation:
(iii) If $B_1$ and $B_2$ are a partition of $\Omega$, then:

$$P(A) = P(A | B_1) P(B_1) + P(A | B_2) P(B_2)$$

Using this formula, we get

$$P(L) = P(L | T) P(T) + P(L | A) P(A)$$

$$= (0.21)(0.29) + (0.14)(0.71) = 0.16$$

**Problem 5 (Bayes’ Theorem):**

**Step 1.** Define the events:

- $T =$ teenagers
- $A =$ adults
- $L =$ listeners
- $L^c =$ non-listeners
- $\Omega =$ set of all people

Note that $T \cup A = \Omega$ and $T \cap A = \emptyset$. Hence $T$ and $A$ form a partition for $\Omega$.

**Step 2.** Write down what you know. Given:

- $P(T) = 0.29$ (probability of randomly selecting a teen)
- $P(A) = 0.71$ (probability of randomly selecting an adult)

Advertising does not change the number of teens and adults in the population. Hence $P(T)$ and $P(A)$ remain the same under both plans. Advertising does change the number of listeners; so we’ll use $L_1$ and $L_2$ for the sets of listeners under plans 1 and 2.

**Plan 1:**
- $P(L_1 | A) = 0.17$ (number of adult listeners increases)
- $P(L_1 | T) = 0.21$ (number of teen listeners remains the same)

The fraction of listeners $L_1$ under plan 1 is:

$$P(L_1) = P(L_1 | A) P(A) + P(L_1 | T) P(T)$$

$$= (0.17)(0.71) + (0.21)(0.29)$$

$$= 0.1816$$

**Plan 2:**
- $P(L_2 | A) = 0.14$ (number of adult listeners remains the same)
- $P(L_2 | T) = 0.27$ (number of teen listeners increases)
The fraction of listeners $L_2$ under plan 2 is:

$$P(L_2) = P(L_2 \mid A)P(A) + P(L_2 \mid T)P(T)$$
$$= (0.14)(0.71) + (0.27)(0.29)$$
$$= 0.1777$$

(i) Since $P(L_1) > P(L_2)$, we’d expect a larger fraction of listeners under plan 1.

(ii) Under plan 1, roughly 18.16% of the population would listen to your station.

Problem 6 (Bayes’ Theorem):
Step 1: Define the events:
- $S$ = set of people with strong form
- $M$ = set of people with mild form
- $F$ = set of people who are disease-free
- $\Omega$ = entire population.
- $N$ = set of people who test negative
- $N^c$ = set of people who test positive

Since $\Omega = S \cup M \cup F$, $S \cap M = \emptyset$, $S \cap F = \emptyset$, $M \cap F = \emptyset$, the collection of sets $S$, $M$, and $F$ forms a partition of the population $\Omega$.

Step 2: Write down the given information:
- $P(S) = 0.08$ (probability that a person has the strong form)
- $P(M) = 0.17$ (probability that a person has the mild form)
- $P(F) = 0.75$ (probability that a person is disease-free)
- $P(N \mid S) = 0.05$ \quad $P(N^c \mid S) = 1 - P(N \mid S) = 0.95$
- $P(N \mid M) = 0.13$ \quad $P(N^c \mid M) = 1 - P(N \mid M) = 0.87$
- $P(N^c \mid F) = 0.11$ \quad $P(N \mid F) = 1 - P(N^c \mid F) = 0.89$

(i) What percent of adults test positive?

$$P(N^c) = P(N^c \mid S)P(S) + P(N^c \mid M)P(M) + P(N^c \mid F)P(F)$$
$$= (.95)(.08) + (.87)(.17) + (.11)(.75) \approx 0.31 = 31\%$$

(ii) What percent of the adult population would you expect to test negative?
Since $N$ and $N^c$ is a partition of $\Omega$,

$$P(N) = 1 - P(N^c) \approx 0.69 = 69\%$$
1.6 Bayes’ theorem

Problem 10 (Bayes’ Theorem):

**Step 1.** Define the events: \( E, F = E^c \), and \( G \). \( E \) and \( E^c \) partition \( \Omega \).

**Step 2.** Write down what you know. Given:

\[
\begin{align*}
\mathcal{P}(E) &= 0.7 \\
\mathcal{P}(E^c) &= 1 - \mathcal{P}(E) = 1 - 0.7 = 0.3 \\
\mathcal{P}(G | E) &= 0.2 \\
\mathcal{P}(G | E^c) &= 0.5
\end{align*}
\]

**Step 3.** Use Bayes’ Theorem:

\[
\begin{align*}
\mathcal{P}(E | G) &= \frac{\mathcal{P}(G | E)\mathcal{P}(E)}{\mathcal{P}(G | E)\mathcal{P}(E) + \mathcal{P}(G | E^c)\mathcal{P}(E^c)} \\
&= \frac{(0.2)(0.7)}{(0.2)(0.7) + (0.5)(0.3)} \\
&= \frac{0.14}{0.14 + 0.15} = \frac{14}{29} \approx 0.483 \approx \left( \frac{1}{2} \right)^-
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}(E^c | G) &= \frac{\mathcal{P}(G | E^c)\mathcal{P}(E^c)}{\mathcal{P}(G | E)\mathcal{P}(E) + \mathcal{P}(G | E^c)\mathcal{P}(E^c)} \\
&= \frac{(0.5)(0.3)}{(0.2)(0.7) + (0.5)(0.3)} \\
&= \frac{0.15}{0.14 + 0.15} = \frac{15}{29} \approx 0.517 \approx \left( \frac{1}{2} \right)^+
\end{align*}
\]
**Problem 11 (Bayes’ Theorem):**
The experiment consists of selecting a person and testing whether or not they have a certain disorder.

**Step 1:** Let

\[ \Omega = \text{all people}, \]
\[ D = \text{event that the person has the disorder}, \]
\[ D^c = \text{event that the person does not have the disorder}, \]
\[ P = \text{event that the person tests positive}, \]
\[ N = \text{event that the person tests negative}. \]

Notice that \( D \) and \( D^c \) partition the set of all people into those who have the disorder and those who don’t \((D \cup D^c = \Omega)\).

**Step 2:** We are given

\[ \mathcal{P}(D) = .07, \]
\[ \mathcal{P}(N|D) = .12 \text{ (probability that someone with the disorder will test negative)} \]
\[ \mathcal{P}(P|D^c) = .15 \text{ (probability that someone without the disorder tests positive)} \]

From this information, we can immediately get three more facts:

\[ \mathcal{P}(D^c) = 1 - \mathcal{P}(D) = .93. \]

If 12% of the people with the disorder test negative, then the other 88% of the people with the disorder test positive: \( \mathcal{P}(P|D) = .88. \)

If 15% of the people without the disorder test positive, then 85% of the people without the disorder test negative: \( \mathcal{P}(N|D^c) = .85 \).

**Step 3:**

In Part \((i)\), we were told that the person tested negative, so that event has already happened. We want \( \mathcal{P}(D|N) \).

In Part \((ii)\), we were told that they had tested positive, so that event has already happened. We want \( \mathcal{P}(D|P) \).

We use Bayes’ theorem.

\[(i) \quad \mathcal{P}(D|N) = \frac{\mathcal{P}(N|D)\mathcal{P}(D)}{\mathcal{P}(N|D)\mathcal{P}(D) + \mathcal{P}(N|D^c)\mathcal{P}(D^c)} = \frac{(.12)(.07)}{(.12)(.07) + (.85)(.93)}\]

\[(ii) \quad \mathcal{P}(D|P) = \frac{\mathcal{P}(P|D)\mathcal{P}(D)}{\mathcal{P}(P|D)\mathcal{P}(D) + \mathcal{P}(P|D^c)\mathcal{P}(D^c)} = \frac{(.88)(.07)}{(.88)(.07) + (.15)(.93)}\]

**Problem 12 (Bayes’ Theorem):**

**Step 1.** Define the events:

\[ \Omega = \text{all people that are adults} \]
\[ S = \text{all people with strong form of disease} \]
\[ M = \text{all people with mild form of disease} \]
\[ F = \text{all people who are free from the disease} \]
The sets \( S \), \( M \), and \( F \) are disjoint, and their union is \( \Omega \). Thus \( S \), \( M \), and \( F \) form a partition of \( \Omega \).

Also, define the events:

\[
T^+ = \text{event of testing positive} \\
T^- = \text{event of testing negative}
\]

The events \( T^+ \) and \( T^- \) are disjoint.

**Step 2.** Write down what you know. Given:

\[
\begin{align*}
\mathcal{P}(S) &= 0.08 \\
\mathcal{P}(M) &= 0.17 \\
\mathcal{P}(F) &= 1 - 0.08 - 0.17 = 0.75 \\
\mathcal{P}(T^- | S) &= 0.05 \quad \text{so} \quad \mathcal{P}(T^+ | S) = 1 - \mathcal{P}(T^- | S) = 0.95 \\
\mathcal{P}(T^- | M) &= 0.13 \quad \text{so} \quad \mathcal{P}(T^+ | M) = 1 - \mathcal{P}(T^- | M) = 0.87 \\
\mathcal{P}(T^+ | F) &= 0.11 \quad \text{so} \quad \mathcal{P}(T^- | F) = 1 - \mathcal{P}(T^+ | F) = 0.89 \\
\end{align*}
\]

**Step 3.** Use Bayes’ Theorem:

\[
\begin{align*}
\mathcal{P}(S | T^+) &= \frac{\mathcal{P}(T^+ | S)\mathcal{P}(S)}{\mathcal{P}(T^+ | S)\mathcal{P}(S) + \mathcal{P}(T^+ | M)\mathcal{P}(M) + \mathcal{P}(T^+ | F)\mathcal{P}(F)} \\
&= \frac{(0.95)(0.08)}{(0.95)(0.08) + (0.87)(0.17) + (0.11)(0.75)} \\
&= 0.248 \\
\mathcal{P}(M | T^+) &= \frac{\mathcal{P}(T^+ | M)\mathcal{P}(M)}{\mathcal{P}(T^+ | S)\mathcal{P}(S) + \mathcal{P}(T^+ | M)\mathcal{P}(M) + \mathcal{P}(T^+ | F)\mathcal{P}(F)} \\
&= \frac{(0.87)(0.17)}{(0.95)(0.08) + (0.87)(0.17) + (0.11)(0.75)} \\
&= 0.483 \\
\mathcal{P}(F | T^-) &= \frac{\mathcal{P}(T^- | F)\mathcal{P}(F)}{\mathcal{P}(T^- | S)\mathcal{P}(S) + \mathcal{P}(T^- | M)\mathcal{P}(M) + \mathcal{P}(T^- | F)\mathcal{P}(F)} \\
&= \frac{(0.89)(0.75)}{(0.05)(0.08) + (0.13)(0.17) + (0.89)(0.75)} \\
&= 0.962
\end{align*}
\]
2 Material for Project 2

2.1 Compound Interest

Problems

Give your answers with monetary amounts rounded to whole dollars, and with rates and yields rounded to 3 decimal places.

Exercise 1 (Compound Interest): (i) Find the future value of $74,000 invested for 3\frac{1}{2} \text{ years at } 5.25\%, \text{ compounded monthly.} \ (ii) \text{ Find the effective annual yield for this investment.}

Exercise 2 (Compound Interest): (i) Find the future value of $150,000 invested for 5 \text{ years at } 6.2\%, \text{ compounded quarterly.} \ (ii) \text{ Find the effective annual yield for this investment.}

Exercise 3 (Compound Interest): (i) Find the the future value of $45,000 invested for 12 \text{ years at } 5.65\%, \text{ compounded daily.} \ (ii) \text{ Find the effective annual yield for this investment.}

Exercise 4 (Compound Interest): What annual rate \( r \), compounded monthly, will have an effective annual yield of 5.25%?

Exercise 5 (Compound Interest): What annual rate \( r \), compounded quarterly, will have an effective annual yield of 4.5%?

Exercise 6 (Compound Interest): What annual rate \( r \), compounded quarterly, will have an effective annual yield of 6.1%?

Exercise 7 (Compound Interest): What annual rate \( r \), compounded annually, will have an effective annual yield of 6.3%?

Exercise 8 (Compound Interest): What annual rate \( r \), compounded quarterly, will have an effective annual yield of 4.39%?

Exercise 9 (Compound Interest): Use Excel to verify that

\[ f(m) = \left( 1 + \frac{1}{m} \right)^m. \]

is an increasing function of \( m \) for the first 1000 values of \( m \).
Exercise 10 (Compound Interest): Use Excel to verify that
\[
\lim_{m \to \infty} \left( 1 + \frac{1}{m} \right)^m = e, \text{ rounded to 4 decimal places.}
\]

Exercise 11 (Compound Interest): (i) Create a new Excel file, similar to the one in the previous problem, that computes values of \((1 + x)^{1/x}\) for positive values of \(x\) that approach 0. (ii) What do you conclude about
\[
\lim_{x \to 0^+} (1 + x)^{1/x}?
\]

Exercise 12 (Compound Interest): (i) Find the future value of $750,000 invested for 3 years and 4 months at a rate of 6.1%, compounded continuously. Round your answer to whole dollars. (ii) Find the effective annual yield (rounded to 3 places) for this investment.

Exercise 13 (Compound Interest): (i) Find the future value of $1,250,000 invested for 12 years and 6 months at a rate of 6.7%, compounded continuously. (ii) Find the effective annual yield for this investment.

Exercise 14 (Compound Interest): Experiment with Excel to approximate the annual rate \(r\) that has a yield of 6% when compounded continuously.

Exercise 15 (Compound Interest): What annual rate \(r\), compounded quarterly, would have the same yield as an annual rate of 6%, compounded continuously?

Exercise 16 (Compound Interest): You invest $2000 at an interest rate of 6.25% compounded daily. How long will it take for this investment to reach $3000? Round your answer to the nearest day.

Exercise 17 (Compound Interest): Redo Exercise 16 with an interest rate of 4.5%.

Exercise 18 (Compound Interest): Find the annual rate \(r\) that produces an effective annual yield of 5.15% when compounded continuously. Round your answer to 3 places.

Exercise 19 (Compound Interest): Find the annual rate \(r\) that produces an effective annual yield of 6.3% when compounded continuously. Round your answer to 3 places.

Exercise 20 (Compound Interest): You invest $40,000 at 5.6% interest, compounded monthly. How long will it take for this investment to reach $65,000? Round to the nearest whole month.

Exercise 21 (Compound Interest): You invest $40,000 at 5.6% interest, compounded continuously. How long will it take for this investment to reach $65,000? Round to the
nearest whole month.

**Exercise 22 (Compound Interest):** You invest $40,000 at 6.0% interest, compounded daily. How long will it take for this investment to reach $80,000? Round to the nearest whole month.

**Exercise 23 (Compound Interest):** What is the present value of a $250,000 payment 6 years from now, if the expected annual rate over this period is 6.15%, compounded continuously? Round your answer to the nearest whole dollar.

**Exercise 24 (Compound Interest):** What is the present value of a $250,000 payment 6 years from now, if the expected annual rate over this period is 6.15%, compounded quarterly? Round your answer to the nearest whole dollar.

**Exercise 25 (Compound Interest):** What is the future value of $25,000 invested for 7\( \frac{1}{2} \) years at an annual rate of 5.75%, compounded continuously? Round your answer to the nearest whole dollar.

**Exercise 26 (Compound Interest):** What is the future value of $25,000 invested for 7\( \frac{1}{2} \) years at an annual rate of 5.75%, compounded daily?

**Exercise 27 (Compound Interest):** A customer is to pay for a $2,149 computer in 48 equal monthly payments, starting with a payment one month after the purchase. If interest is compounded continuously at an annual rate of 6.25%, what is the amount of each payment? What is the total amount of the payments?

**Exercise 28 (Compound Interest):** You want to buy a motorcycle that costs $23,585. It is to be paid for in 36 equal monthly payments, starting with a payment one month after the purchase. If interest is compounded continuously at an annual rate of 9.5%, what is the amount of each payment?

**Exercise 29 (Compound Interest):** Compute the yearly ratio of future to present value that corresponds to an annual rate of 6.25%, compounded continuously. Round your answer to 6 places.

**Exercise 30 (Compound Interest):** Compute the monthly rate of interest, compounded continuously, that corresponds to a monthly ratio of future to present value of 1.002. Round your answer to 6 places.

**Exercise 31 (Compound Interest):** (i) Compute the risk-free weekly rate, \( r_{rf} \), for your team’s risk-free annual rate. (ii) Compute the risk-free weekly ratio, \( R_{rf} \), that corresponds to this rate.
Solutions

NOTE: In the problems involving compound interest, \( F(t) = Pe^{rt} \) (continuous) or \( F(t) = P(1 + \frac{r}{n})^{nt} \) (discrete). To find \( F \), we need \( P \), \( r \), \( n \), and \( t \). Yield is given by \( y = e^r - 1 \) (continuous) or \( y = \left(1 + \frac{r}{n}\right)^n - 1 \) (discrete).

Exercise 1 (Compound Interest): (i) Find the future value of $74,000 invested for 3\frac{1}{2} \text{ years at 5.25\%}, \text{ compounded monthly.} \ (ii) \text{Find the effective annual yield for this investment.}

Solution. Given: \( P = 74,000 \), \( t = 3.5 \), \( r = .0525 \), and \( n = 12 \) (since compounded monthly).

\[
(i) \quad F(3.5) = 74,000 \left(1 + \frac{.0525}{12}\right)^{12(3.5)} \\
(ii) \quad y = \left(1 + \frac{.0525}{12}\right)^{12} - 1
\]

Exercise 2 (Compound Interest): (i) Find the value of $150,000 invested for 5 years at 6.2\% \text{, compounded quarterly.} \ (ii) \text{Find the effective annual yield for this investment.}

Solution. Given: \( P = 150,000 \), \( t = 5 \), \( r = .062 \), and \( n = 4 \) (since compounded quarterly).

\[
(i) \quad F(5) = 150,000 \left(1 + \frac{.062}{4}\right)^{4\times5} \\
(ii) \quad y = \left(1 + \frac{.062}{4}\right)^{4} - 1
\]

Exercise 3 (Compound Interest): (i) Find the value of $45,000 invested for 12 years at 5.65\% \text{, compounded daily.} \ (ii) \text{Find the effective annual yield for this investment.}

Solution. Given: \( P = 45,000 \), \( t = 12 \), \( r = .0565 \), and \( n = 365 \) (since compounded daily).

\[
(i) \quad F(12) = 45,000 \left(1 + \frac{.0565}{365}\right)^{365\times12} \\
(ii) \quad y = \left(1 + \frac{.0565}{365}\right)^{365} - 1
\]
Exercise 4 (Compound Interest): What annual rate \( r \), compounded monthly, will have an effective annual yield of 5.25%?

Solution. Given: \( n = 12, y = .0525 \). Want \( r \).

\[
r = n \left[ (1 + y)^{1/n} - 1 \right] = 12 \left[ (1 + .0525)^{1/12} - 1 \right]
\]

Exercise 5 (Compound Interest): What annual rate \( r \), compounded quarterly, will have an effective annual yield of 4.5%?

Solution. Given: \( n = 4, y = .045 \). Want \( r \).

\[
r = 4 \left[ (1 + .045)^{1/4} - 1 \right]
\]

Exercise 6 (Compound Interest): What annual rate \( r \), compounded quarterly, will have an effective annual yield of 6.1%?

Solution. Given: \( n = 4, y = .061 \). Want \( r \).

\[
r = 4 \left[ (1 + .061)^{1/4} - 1 \right]
\]

Exercise 7 (Compound Interest): What annual rate \( r \), compounded annually, will have an effective annual yield of 6.3%?

Solution. Given: \( n = 1 \) (since compounded yearly); \( y = .063 \). Want \( r \).

\[
r = \left[ (1 + .063)^{1/1} - 1 \right] = \left[ (1 + .063) - 1 \right] = .063 = y
\]

Notice that when \( n = 1 \), it’s always true that \( r = y \).

Exercise 8 (Compound Interest): What annual rate \( r \), compounded quarterly, will have an effective annual yield of 4.39%?

Solution. Given: \( n = 4, y = .0439 \). Want \( r \).

\[
r = 4 \left[ (1 + .0439)^{1/4} - 1 \right]
\]

Exercise 9 (Compound Interest): Use Excel to verify that

\[ f(m) = \left( 1 + \frac{1}{m} \right)^m \]

is an increasing function of \( m \) for the first 1000 values of \( m \).

Exercise 10 (Compound Interest): Use Excel to verify that
\[
\lim_{m \to \infty} \left( 1 + \frac{1}{m} \right)^m = e, \text{ rounded to 4 decimal places.}
\]


Exercise 11 (Compound Interest): (i) Create a new Excel file, similar to the one in the previous problem, that computes values of \((1 + x)^{1/x}\) for positive values of \(x\) that approach 0. (ii) What do you conclude about
\[
\lim_{x \to 0^+} (1 + x)^{1/x}?
\]

Solution. The answer is \(e\).

Exercise 12 (Compound Interest): (i) Find the value of $750,000 invested for 3 years and 4 months at a rate of 6.1%, compounded continuously. Rounded your answer to whole dollars. (ii) Find the effective annual yield (rounded to 3 places) for this investment.

Solution. Given: \(P = 750,000, t = 3 \frac{1}{3} \approx 3.333\) years, \(r = .061\).

\[
(i) \quad F = 750,000e^{(.061)(3.333)}
\]

\[
(ii) \quad y = e^r - 1 = e^{.061} - 1
\]

Exercise 13 (Compound Interest): (i) Find the value of $1,250,000 invested for 12 years and 6 months at a rate of 6.7%, compounded continuously. (ii) Find the effective annual yield for this investment.

Solution. Given: \(P = 1,250,000, t = 12.5, \text{ and } r = .067\).

\[
(i) \quad F = 1,250,000e^{(.067)(12.5)}
\]

\[
(ii) \quad y = e^r - 1 = e^{.067} - 1
\]

Exercise 14 (Compound Interest): Experiment with Excel to approximate the annual rate \(r\) that has a yield of 6% when compounded continuously.

Solution. Given \(y = .06\), want \(r\). Step 1: solve for \(r\).

\[
y = e^r - 1 \quad \xrightarrow{+1} \quad e^r = y + 1 \quad \xrightarrow{\ln()} \quad \ln[e^r] = r = \ln(1 + y).
\]

Step 2: substitute the known values.

\[
r = \ln(1 + y) = \ln(1 + .06) = \ln(1.06)
\]
We can get an approximate answer without using Excel or a calculator. For small $y$, 
$\ln(1 + y) \approx y$, so $r \approx y$. To be precise, solve $r = \ln(1 + y)$ with $y = .06$.

**Exercise 15 (Compound Interest):** What annual rate $r$, compounded quarterly, would have the same yield as an annual rate of 6%, compounded continuously?

**Solution.** Want $r$ so that 
$$P \left(1 + \frac{r}{n}\right)^n = P(1 + y) = Pe^{.06}. $$

Put another way, we want to solve for $r$ in the equation 
$$\left(1 + \frac{r}{n}\right)^n - 1 = y = e^{.06} - 1.$$  
Cancelling 1 from both sides gives $(1 + \frac{r}{n})^n = e^{.06}$. Since we compound quarterly, $n = 4$. The equation becomes $(1 + \frac{r}{4})^4 = e^{.06}$. Taking the 4th root gives 
$$1 + \frac{r}{4} = (e^{.06})^{1/4} = e^{.06/4} = e^{.015}.$$  
Solving for $r$ yields 
$$r = 4 \left[e^{.015} - 1 \right].$$

**Exercise 16 (Compound Interest):** You invest $2000 at an interest rate of 6.25%, compounded daily. How long will it take for this investment to reach $3000? Round your answer to the nearest day.

**Solution.** Given: $P = 2000, F = 3000, r = .0625$, and $n = 365$. Solve for $t$. 

$$F = P \left(1 + \frac{r}{n}\right)^{nt} \quad \overset{\div P}{\Rightarrow} \quad \left(1 + \frac{r}{n}\right)^{nt} = \frac{F}{P} \quad \overset{\ln( \cdot )}{\Rightarrow} \quad \ln \left(1 + \frac{r}{n}\right)^{nt} = \ln \left(\frac{F}{P}\right) \quad \Rightarrow \quad nt \ln \left(1 + \frac{r}{n}\right) = \ln \left(\frac{F}{P}\right) \quad \text{(property of logarithms)}$$  

$$\Rightarrow \quad t = \frac{\ln(F/P)}{n \ln \left(1 + \frac{r}{n}\right)} \quad \text{(General formula for } t)$$

For our problem 
$$t = \frac{\ln(3000/2000)}{365 \ln \left(1 + \frac{.0625}{365}\right)}. $$

Round this solution to the nearest whole day.
Exercise 17 (Compound Interest): Redo Exercise 16 with an interest rate of 4.5%.

Solution. This is just like Exercise 16. We solve it exactly the same way; the only change we make is substituting \( r = 0.045 \) for \( r = 0.0625 \).

Exercise 18 (Compound Interest): Find the annual rate \( r \) that produces an effective annual yield of 5.15% when compounded continuously. Round your answer to 3 places.

Solution. Given: compounded continuously, \( y = 0.0515 \). Want \( r \).

\[
r = \ln(1 + y) = \ln(1 + 0.0515) = \ln(1.0515)
\]

Exercise 19 (Compound Interest): Find the annual rate \( r \) that produces an effective annual yield of 6.3% when compounded continuously. Round your answer to 3 places.

Solution. Given: compounded continuously, \( y = 0.063 \). Find \( r \).

\[
r = \ln(1 + 0.063) = \ln(1.063)
\]

Exercise 20 (Compound Interest): You invest $40,000 at 5.6% interest, compounded monthly. How long will it take for this investment to reach $65,000? Round to the nearest whole month.

Solution. Given: \( P = 40,000 \), \( F = 65,000 \), \( r = 0.056 \), and \( n = 12 \) (compounded monthly). Want \( t \). We can use the general formula we derived in the solution for Exercise 16:

\[
t = \frac{\ln(F/P)}{n \ln (1 + \frac{r}{n})} = \frac{\ln(65/40)}{12 \ln (1 + \frac{0.056}{12})}
\]

Exercise 21 (Compound Interest): You invest $40,000 at 5.6% interest, compounded continuously. How long will it take for this investment to reach $65,000? Round to the nearest whole month.

Solution. Given: compounded continuously, \( P = 40,000 \), \( F = 65,000 \), \( r = 0.056 \). Find \( t \) to the nearest month.

\[
F = Pe^{rt} \quad \rightarrow \quad e^{rt} = \frac{F}{P} \quad \ln() \quad \ln(e^{rt}) = \ln \left( \frac{F}{P} \right) \quad \text{P.O.L.} \quad rt = \ln \left( \frac{F}{P} \right)
\]

Solving for \( t \):

\[
t = \frac{1}{r} \ln \left( \frac{F}{P} \right),
\]
where P.O.L. = property of logarithms. For our problem, we want
\[
t = \frac{1}{.056} \ln \left( \frac{65,000}{40,000} \right) = 8.66978 \text{ years}
\]
Remember that this gives us \( t \) in years. The problem asks us to round to the nearest month, or \( 1/12 \) year. We must convert our answer, expressed in years, into months:
\[
8.66978 \text{ years} = (8.66978)(12) \text{ months} \approx 104 \text{ months}.
\]

**Exercise 22 (Compound Interest):** You invest \$40,000 at 6.0% interest, compounded daily. How long will it take for this investment to reach \$80,000? Round to the nearest whole month.

**Solution.** Given \( P = 40,000, F = 80,000 \) (so \( F/P = 2 \)), \( r = .06 \), and \( n = 365 \) (compounded daily). Want \( t \), to the nearest whole month.
This is very similar to Exercises 16 and 20, and we’ll use the same general formula:
\[
t = \frac{\ln(F/P)}{n \ln(1 + \frac{r}{n})} = \frac{\ln(2)}{365 \ln (1 + \frac{.06}{365})}
\]

**Exercise 23 (Compound Interest):** What is the present value of a \$250,000 payment 6 years from now, if the expected annual rate over this period is 6.15%, compounded continuously? Round your answer to the nearest whole dollar.

**Solution.** Given: Compounded continuously, \( F = 250,000, r = .0615 \), and \( t = 6 \). Want \( P \).
\[
P = Fe^{-rt} = 250,000e^{- (.0615)6}
\]

**Exercise 24 (Compound Interest):** What is the present value of a \$250,000 payment 6 years from now, if the expected annual rate over this period is 6.15%, compounded quarterly? Round your answer to the nearest whole dollar.

**Solution.** Given: \( F = 250,000, r = .0615, n = 4 \) (compounded quarterly), and \( t = 6 \). Want \( P \).
\[
P = F \left( 1 + \frac{r}{n} \right)^{-nt} = 250,000 \left( 1 + \frac{.0615}{4} \right)^{-46}
\]
Exercise 25 (Compound Interest): What is the future value of $25,000 invested for \(7\frac{1}{2}\) years at an annual rate of 5.75\%, compounded continuously? Round your answer to the nearest whole dollar.

**Solution.** Given: Compounded continuously, \(P = 25,000\), \(r = .0575\), and \(t = 7.5\). Find \(F\).

\[
F = Pe^{rt} = 25,000e^{(.0575)(7.5)}
\]

Exercise 26 (Compound Interest): What is the future value of $25,000 invested for \(7\frac{1}{2}\) years at an annual rate of 5.75\%, compounded daily?

**Solution.** Given: \(n = 365\) (compounded daily), \(P = 25,000\), \(t = 7.5\), and \(r = .0575\). Find \(F\).

\[
F = P \left(1 + \frac{r}{n}\right)^{nt} = 25,000 \left(1 + \frac{.0575}{365}\right)^{(365)(7.5)}
\]

Exercise 27 (Compound Interest): A customer is to pay for a $2,149 computer in 48 equal monthly payments, starting with a payment one month after the purchase. If interest is compounded continuously at an annual rate of 6.25\%, what is the amount of each payment? What is the total amount of the payments?

**Solution.** In example 8 in the Thompson book, replace 24 with 48 and repeat the calculation.
Exercise 28 (Compound Interest): You want to buy a motorcycle that costs $23,585. It is to be paid for in 36 equal monthly payments, starting with a payment one month after the purchase. If interest is compounded continuously at an annual rate of 9.5%, what is the amount of each payment?

Solution. Given: Compounded continuously, \( P = 23,585 \), \( r = .095 \). The payments will be \( F_1, \ldots, F_{36} \), where the \( i \)th payment \( F_i \) will be made \( i \) months, or \( i/12 \) years from now. The present value of \( F_i \) is \( P_i = F_i e^{-r(i/12)} \). Thus

\[
P = \sum_{i=1}^{36} P_i = \sum_{i=1}^{36} F_i e^{-r(i/12)}
\]

Since all payments \( F_i \) are equal, we can call that amount \( F = F_1 = F_2 = \cdots = F_{36} \)

\[
= F \sum_{i=1}^{36} e^{-r(i/12)}
= F \sum_{i=1}^{36} \left[e^{-r/12}\right]^i
\]

Factor out \( e^{-r/12} \) and re-order the sum \( (k = i - 1) \) to get the geometric series

\[
= F e^{-r/12} \sum_{k=0}^{35} \left[e^{-r/12}\right]^k = F e^{-r/12} \left[\frac{1 - \left(e^{-r/12}\right)^{36}}{1 - e^{-r/12}}\right] = F e^{-r/12} \left[\frac{1 - e^{-3r}}{1 - e^{-r/12}}\right]
\]

Then

\[
F = P \left(e^{-r/12} \left[\frac{1 - e^{-3r}}{1 - e^{-r/12}}\right]\right)^{-1} = P e^{r/12} \left[\frac{1 - e^{-r/12}}{1 - e^{-3r}}\right].
\]

Finish by substituting the given values for the parameters.
Exercise 29 (Compound Interest): Compute the yearly ratio of future to present value that corresponds to an annual rate of 6.25%, compounded continuously. Round your answer to 6 places.

Solution. Given: Compounded continuously, \( r = .0625 \). We want a yearly ratio \( R \):

\[
R \equiv \frac{F}{P} = e^r.
\]

(For \( t = 1 \) year, \( F(t)/P = e^{rt} = e^r \).) For \( r = .0625 \), we have \( R = e^{.0625} \).

Exercise 30 (Compound Interest): Compute the monthly rate of interest, compounded continuously, that corresponds to a monthly ratio of future to present value of 1.002. Round your answer to 6 places.

Solution. Given: compounded continuously, \( R = 1.002 \). Want \( r \).

\[
r = \ln(R) = \ln(1.002).
\]

Exercise 31 (Compound Interest): (i) Compute the risk-free weekly rate, \( r_{rf} \), for your team’s risk-free annual rate. (ii) Compute the risk-free weekly ratio, \( R_{rf} \), that corresponds to this rate.

Solution. Results will vary depending on your team’s Project 2 assignment.
2.2 Finite Probability Distributions

Exercise 4 (Probability Distribution):
Let $X$ be a finite random variable with the following probability mass function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_X(x)$</td>
<td>.1</td>
<td>.3</td>
<td>.4</td>
<td>.2</td>
</tr>
</tbody>
</table>

Plot a bar graph of $f_X(x)$.

Exercise 5 (Probability Distribution):
Compute $F_X(x)$ for the $f_X(x)$ given in Exercise 4.

Since 1 is the smallest possible value for $X$,

$$F_X(x) = \mathbb{P}(X \leq x) = 0 \quad \text{for } x < 1.$$  

At $x = 1$,

$$F_X(1) = f_X(1) = \mathbb{P}(X = 1) = \frac{1}{10}.$$

For $1 \leq x < 2$,

$$F_X(x) = F_X(1) = \frac{1}{10}.$$

For $2 \leq x < 3$,

$$F_X(x) = f_X(1) + f_X(2) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10}.$$

For $3 \leq x < 4$,

$$F_X(x) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) = \frac{1}{10} + \frac{3}{10} + \frac{4}{10} = \frac{8}{10}.$$

For $4 \leq x$,

$$F_X(x) = \sum_{i=1}^{4} \mathbb{P}(X = i) = 1.$$
Here’s a graph of $F_X(x)$ versus $x$:

![Graph of $F_X(x)$ versus $x$]

Exercise 6 (Probability Distribution):
Let $X$ be a random variable with the following cumulative distribution function:

\[
F_X(x) = \begin{cases} 
1 & \text{if } x > 8 \\
\frac{7}{10} & \text{if } 6 \leq x < 8 \\
\frac{3}{10} & \text{if } 4 \leq x < 6 \\
0 & \text{if } x < 4.
\end{cases}
\]

(i) What are the possible values for $X$?

The points of discontinuity in the graph of $F_X(x)$ will be the values in the range of $X$. $F_X(x)$ has discontinuities at $x = 4, 6, 8$. Thus the range of $X$ is $\{4, 6, 8\}$.

(ii) Find all values for $f_X(x)$.

We will use property (v) on page ??:

\[
f_X(x_i) = F_X(x_i) \\
f_X(x_i) = F_X(x_i) - F_X(x_{i-1}) \quad \text{for } i = 2, \ldots, N.
\]

Here $x_1 = 4$, $x_2 = 6$, and $x_3 = 8$. We get

\[
f_X(4) = F_X(4) = \frac{3}{10}, \\
f_X(6) = F_X(6) - F_X(4) = \frac{7}{10} - \frac{3}{10} = \frac{4}{10}, \\
f_X(8) = F_X(8) - F_X(6) = 1 - \frac{7}{10} = \frac{3}{10}.
\]

As a check, notice that

\[
\sum_{all \ x} f_X(x) = f_X(4) + f_X(6) + f_X(8) = 1 = \frac{3}{10} + \frac{4}{10} + \frac{3}{10} = 1.
\]
2.3 Continuous Probability Distribution

Exercises 18 and 19 (Probability Distributions):

Using $a = 0$ and $b = 120$ in Equations (??) for $f_X(x)$ and (??) for $F_X(x)$ gives us

\[
f_X(x) = \begin{cases} 
\frac{1}{120} & \text{if } 0 \leq x \leq 120 \\
0 & \text{if } x \notin [0, 120]
\end{cases}
\]

\[
F_X(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x}{120} & \text{if } 0 \leq x \leq 120 \\
1 & \text{if } x > 120
\end{cases}
\]

For exercise 18, we want the probability that $40 \leq X \leq 85$. We’ll use the formula for $F_X(x)$:

\[
\mathcal{P}(40 \leq X \leq 85) = F_X(85) - F_X(40) = \frac{85 - 40}{120} = \frac{45}{120}.
\]

For exercise 19, we need the probability that the can falls off in the first 32 feet:

\[
\mathcal{P}(0 \leq X \leq 32) = \mathcal{P}(X \leq 32) = F_X(32) = \frac{32}{120}.
\]

**Exercise 24 (Probability Distribution):**

From Example 5, we have an exponential random variable with $\alpha = 2$. We want to compute the probability that the time between the arrival of consecutive bank customers is at most 1 minute 40 seconds. Since that is 5/3 minutes, we need $\mathcal{P}(X \leq 5/3)$.

\[
\mathcal{P}(X \leq 5/3) = 1 - e^{-\left(\frac{5}{3} \cdot \frac{1}{2}\right)} = 1 - e^{-5/6} \approx 0.5654.
\]
Exercise 25 (Probability Distribution):
Same as Exercise 24, but since 3 minutes 15 seconds is $3\frac{1}{4}$ minutes, we want
\[ \mathcal{P}(13/4 \leq X) = 1 - \mathcal{P}(X < 13/4) = 1 - \mathcal{P}(X \leq 13/4) = 1 - F_X(13/4) \]
\[ = 1 - \left(1 - e^{-\left(\frac{13}{4}\right)}\right) = e^{-13/8} \approx .1969. \]

Exercise 26 (Probability Distribution):
We’re given that $X$ is an exponential random variable with $\alpha = 3.5$.
(i) $f_X(2) = \frac{1}{3.5} e^{-2/3.5} \approx .1613$.
(ii) $F_X(2) = 1 - e^{-2/3.5} \approx .4353$.

Exercise 27 (Probability Distribution):
We’re given that $X$ is an exponential random variable with $\alpha = 3.5$.
(i) The probability that $X$ takes at most 4 is
\[ \mathcal{P}(X \leq 4) = F_X(4) = 1 - e^{-4/3.5} \approx .6811. \]
(ii) The probability that $X$ takes at least 4 is
\[ \mathcal{P}(4 \leq X) = \mathcal{P}(4 < X) = 1 - \mathcal{P}(X \leq 4) = 1 - F_X(4) \]
\[ = 1 - \left(1 - e^{-4/3.5}\right) = e^{-4/3.5} \approx .3189. \]

Exercise 28 (Probability Distribution):
Here $T$ is the lifetime, in hours, of the graphics card. We’re given that $T$ is an exponential random variable with $\alpha = 5000$.
(i) The probability that the card fails before 3000 hours is
\[ \mathcal{P}(T < 3000) = \mathcal{P}(T \leq 3000) = F_T(3000) = 1 - e^{-3000/5000} = 1 - e^{-3/5} \approx .4512. \]
(ii) The probability that the card lasts at least 5000 hours is
\[ \mathcal{P}(5000 < T) = 1 - \mathcal{P}(T \leq 5000) = 1 - F_T(5000) \]
\[ = 1 - \left(1 - e^{-5000/5000}\right) = e^{-1} \approx .3679. \]
(iii) $\mathcal{P}(T = 5000) = 0$. For any continuous random variable $X$, and for any constant value $x$, $\mathcal{P}(X = x) = 0$.

Exercise 29 (Probability Distribution):
We are asked to find $t_0$ such that $\mathcal{P}(T \leq t_0) = 1/2$. We know that
\[ \mathcal{P}(T \leq t_0) = F_T(t_0) = 1 - e^{-t_0/5000}. \]
Equating the two expressions yields
\[ 1 - e^{-t_0/5000} = \frac{1}{2} \Rightarrow e^{-t_0/5000} = \frac{1}{2}. \]
Take the natural logarithm of both sides of the equation:

\[ \ln \left( e^{-t_0/5000} \right) = \frac{-t_0}{5000} \ln e = \ln \left( \frac{1}{2} \right). \]

We’ve used the property of logarithms: \( \log_b a^x = x \log_b a \). Next, we’ll use the fact that \( \ln e = 1 \), and we’ll solve for \( t_0 \):

\[ t_0 = -5000 \ln \left( \frac{1}{2} \right) = 5000 \ln(2) \approx 3466 \text{ hours}. \]

**Exercise 33 (Probability Distribution):**
The graph appears to be symmetric about \( x = 4 \). Thus \( \mathcal{E}(X) \approx 4 \). If the graph were made of sheet metal, it would balance on a knife edge that ran along the line of symmetry \( x = 4 \).

**Exercise 34 (Probability Distribution):**
We are given \( f_X(x) = .25e^{-.25x} = \frac{1}{4}e^{-x/4} \). We want \( \mathcal{E}(X) \).
For the exponential distribution, \( \mathcal{E}(X) = \alpha \). By comparing the problem’s formula for \( f_X(x) \) with Equation (??), we see that \( \alpha = 4 \). Thus \( \mathcal{E}(X) = 4 \).

**Exercise 35 (Probability Distribution):**
We’re given that \( T \) is an exponential random variable giving the time in months before your firewire card fails. We know that \( \mathcal{E}(T) = 62 \).
(i) In order to find \( F_T(t) \), we need to know the parameter \( \alpha \). But for an exponential random variable, \( \mathcal{E}(T) = \alpha \); so \( \alpha = 62 \). Hence

\[ F_T(t) = 1 - e^{-t/62}. \]

(ii) We want \( P(T \leq 4 \text{ years}) \). Since our formula uses time in months, we need to convert: 4 years = 48 months. Then

\[ P(T \leq 48) = F_T(48) = 1 - e^{-48/62} \approx .5389. \]

**Exercise 3 (Random sampling):**
Here \( S \) is a random variable giving the starting salary of a randomly selected person in your field.
(i) \( \mu_S \approx \frac{1}{10} \left( 31,600 + 22,900 + 28,200 + 28,500 + 34,000 + 30,000 + 25,900 + 26,400 + 27,100 + 29,700 \right) \approx 28,430. \)

On average, you should expect a starting salary of $28,430.