Problem Set 1: Interest theory

1. (Compound Interest) You have $10,000 to invest. You are considering two investment schemes, each offering an annual interest rate of 5%. Scheme A offers 5% interest compounded continuously, while scheme B offers 5% interest compounded quarterly. Which investment scheme will result in the largest future value of your principal?

Solution: Choose scheme A. Recall: if $P$, $r$, and $t$ are fixed, then your investment will grow the fastest under continuous compound interest. This scheme will give you the largest future value.

2. (Compound Interest) Below are two graphs of the future value of the principal $P = $1000 as a function of time. The interest rate for both graphs is $r = .04$. One graph grows at a simple interest rate, while the other grows at a compound interest rate with $n = 12$. Both graphs are plotted over the time interval starting at time $t = 0$ and ending at time $t = 30$ years. Clearly label which one is the graph of simple interest growth. Warning: If I can’t tell which graph you indicate you will receive a zero for the problem.

![Graphs of Future Value](image)

Solution: Recall: An investment of $P$ dollars under a simple interest rate $r$ grows linearly in time. Hence, the graph of simple interest is linear. Moreover, for fixed $P$ and $r$, it is the slowest-growing interest scheme in time. The lower linear graph is the simple interest graph.
3. **(Compound Interest: Future Value)** What is the value of $100 after 10 years at 5% interest, compounded monthly? Round to the nearest dollar.

(a) 1056  (c) 569  (e) 730  (g) 297  (i) 953  (k) 9394  
(b) 723  (d)* 165  (f) 704  (h) 367  (j) 1  (l) none of the these  

**Solution:** Given: \( P = 100, \ t = 10, \ r = .05, \) and \( n = 12. \) Want: \( F. \)  
The discrete interest formula is \( F = P \left(1 + \frac{r}{n}\right)^{nt}. \) Upon substituting the given values into the formula we arrive at \( F = 100 \left(1 + \frac{.05}{12}\right)^{12(10)} = 164.7 \approx 165. \)

4. **(Compound Interest: yield)** What is the effective annual yield \( y \) for problem 3? (Use 3 decimal place accuracy).

(a) .307  (c) .517  (e) .023  (g)* .051  (i) .027  (k) .039  
(b) .024  (d) .039  (f) .105  (h) .002  (j) .095  (l) none of the these  

**Solution:** The formula for effective annual yield in the case of discrete compounding is \( y = \left(1 + \frac{r}{n}\right)^n - 1. \) Substituting the values from problem 3 yields \( y = \left(1 + \frac{.05}{12}\right)^{12} - 1 \approx .051. \)

5. **(Compound Interest: Future Value)** What is the value of $100 after 10 years at 4%, compounded quarterly (rounded to the nearest dollar)?

(b) 1013  (c) 117  (e) 203  (g) 17  (i) 221  (k) 155  
(b) 257  (d) 154  (f)* 149  (h) 196  (j) 6  (l) none of the these  

**Solution:** Given: \( P = 100, \ t = 10, \ r = .04, \) and \( n = 4. \) Want: \( F. \)  
The discrete interest formula is \( F = P \left(1 + \frac{r}{n}\right)^{nt}. \) Upon substituting the given values into the formula we arrive at \( F = 100 \left(1 + \frac{.04}{4}\right)^{4(10)} = 148.9 \approx 149. \)

6. **(Compound Interest: yield)** What is the effective annual yield \( y \) for problem 5?  

(b) 0  (c) .5957  (e) .6579  (g)* .0406  (i) .1106  (k) 1  
(b) .2535  (d) .3676  (f) .6903  (h) .0824  (j) .9068  (l) none of the these  

**Solution:** The formula for effective annual yield in the case of discrete compounding is \( y = \left(1 + \frac{r}{n}\right)^n - 1. \) Substituting the values from problem 5 yields \( y = \left(1 + \frac{.04}{4}\right)^{4} - 1 \approx .04. \)
7. **(Compound Interest: Future Value)** What is the value of $1 after 100 years at 10% interest, compounded continuously? Round to the nearest dollar.

(c) 10567 (c) 8569 (e) 48730 (g) 897 (i) 4953 (k) 93904

(b) 900 (d)* 22026 (f) 7094 (h) 33367 (j) 1 (l) none of these

**Solution:** Given: \(P = 1, \ t = 100, \ r = .1,\) and the compounding scheme is continuous. Want: \(F.\) The continuous compound interest formula is \(F = Pe^{rt}.\) Upon substituting the given values into the formula, we arrive at \(F = 1e^{.1(100)} = e^{10} \approx 22026.47.\)

8. **(Compound Interest: yield)** What is the effective annual yield \(y\) for problem 7? (Use 3 decimal place accuracy).

(c) .307 (c) .567 (e) .023 (g) .490 (i) .325 (k) .639

(b) .924 (d) .739 (f)* .105 (h) .002 (j) .195 (l) none of these

**Solution:** The formula for effective annual yield in the case of continuous compounding is \(y = e^r - 1.\) Substituting the values from problem 5 yields \(y = e^{.1} - 1 \approx 0.105.\)

9. **(Compound Interest: Future Value)** Find the value of $1,250,000 after 12 years and 6 months, if it is invested at a rate of \(\frac{1}{8}\%\) compounded continuously.

**Solution:** Given: \(P = \$1,250,000, \ r = \frac{41}{(8\times100)} = 0.05125, \ t = 12.5\) years.

\[
F = Pe^{rt}
\]
\[
= 1250000e^{(0.05125\times12.5)}
\]
\[
= 2,372,083.19
\]

10. **(Compound Interest: yield)** What is the yield on this investment? Keep your answer accurate to 0.00000001%. You may want to use Excel to compute this.

\[
y = e^r - 1
\]
\[
= e^{(0.05125)} - 1
\]
\[
= 5.25860069\%
\]
11. (Compound Interest: time to double investment) You want to invest P dollars. How long will it take to double your investment at an annual interest rate of 10%, compounded continuously? (round your answer to the nearest year).

(a) 1 (c) 3 (e) 5 (g)* 7 (i) 9 (k) 11
(b) 2 (d) 4 (f) 6 (h) 8 (j) 10 (l) none of the these

**Solution:** Given: \( F = 2P, \ r = .10, \) and compounding scheme is continuous. Want: \( t_{\text{double}}. \)

Substituting these values into the compound interest formula \( F = Pe^{rt} \) yields

\[
2P = Pe^{rt} \Rightarrow 2 = e^{rt} \quad \text{(divide by P)}
\]

\[
\Rightarrow \ln(2) = rt \quad \text{(Take the natural log of both sides)}
\]

\[
\Rightarrow t = \frac{\ln(2)}{r} \approx \frac{.69}{.1} = 6.9 \quad \text{(Solve for t)}
\]

12. (Compound Interest: time to double investment) You want to invest P dollars. How long will it take to double your original investment at an annual interest rate of 6.9%, compounded continuously? Round your answer to the nearest year.

(b) 8 (c)* 10 (e) 12 (g) 14 (i) 16 (k) 18
(b) 9 (d) 11 (f) 13 (h) 15 (j) 17 (l) none of the these

**Solution:** Given: \( F = 2P, \ r = .069, \) and compounding scheme is continuous. Want: \( t. \)

Substituting these values into the compound interest formula \( F = Pe^{rt} \) yields

\[
2P = Pe^{rt} \Rightarrow 2 = e^{rt} \quad \text{(divide by P)}
\]

\[
\Rightarrow \ln(2) = rt \quad \text{(Take the natural log of both sides)}
\]

\[
\Rightarrow t = \frac{\ln(2)}{r} \approx \frac{.69}{.069} = 10 \quad \text{(Solve for t)}
\]
13. (Compound Interest: time to double investment) You want to invest P dollars. How long will it take to double your original investment at an annual interest rate of 4%, compounded quarterly? (round your answer to the nearest year).

(c) 10 (c) 12 (e) 14 (g) 16 (i) 18 (k) 20
(b) 11 (d) 13 (f) 15 (h) 17 (j) 19 (l) none of the these


Substituting these values into the compound interest formula $F = Pe^{rt}$ yields

\[ 2P = P \left(1 + \frac{r}{n}\right)^n \Rightarrow 2 = \left(1 + \frac{r}{n}\right)^n \] (divide by P)

\[ \Rightarrow \ln(2) = nt \ln \left(1 + \frac{r}{n}\right) \] (Take the natural log of both sides)

\[ \Rightarrow t = \frac{\ln(2)}{n \ln \left(1 + \frac{r}{n}\right)} \approx \frac{\ln(2)}{r} \approx \frac{.69}{.04} = 17 \] (Solve for $t$)

14. (Compound Interest) What annual rate, $r$, compounded continuously, would have the same yield as an annual rate of 6%, compounded weekly? Round your answer to the nearest percent (2 decimal place accuracy).

(a) .02 (c) .04 (e)* .06 (g) .08 (i) .10 (k) .12
(b) .03 (d) .05 (f) .07 (h) .09 (j) .11 (l) none of the these

Solution: Given: $r = .06$, $n = 52$ (discrete case). Want: $r$ for the compound interest scheme.

The relevant formulas for yield are

\[
\begin{align*}
y & = e^r - 1 \quad \text{(continuous compounding case)} \\
y & = \left(1 + \frac{r}{n}\right)^n - 1 \quad \text{(discrete compounding case)}
\end{align*}
\]

Equating the two expressions for yield gives

\[ e^r - 1 = y = \left(1 + \frac{.06}{52}\right)^{52} - 1 \Rightarrow e^r = \left(1 + \frac{.06}{52}\right)^{52} \]

\[ \Rightarrow r = \ln \left(1 + \frac{.06}{52}\right)^{52} = 52 \ln \left(1 + \frac{.06}{52}\right) \approx 52 \left( \frac{.06}{52} \right) = .06 \]
Fast way: Recall: \( y \approx r \) for both the discrete and continuous case provided that \( r < 0.2 \).

15. (Compound Interest) Find the annual rate, \( r \), which produces an effective annual yield of 3.75\%, when compounded continuously. Round your answer to the nearest tenth of a percent (three decimal places).

**Solution:** Given: \( y = 0.0375 \) and the yield formula \( y = e^r - 1 \) for continuous compounding.

Want \( r \). We now solve for \( r \).

\[
y = e^r - 1
\]

\[
0.0375 = e^r - 1
\]

\[
1.0375 = e^r \quad \text{(add 1 to both sides)}
\]

\[
\ln(1.0375) = \ln e^r = r \ln e \quad \text{(take the log of both sides)}
\]

\[
3.68\% = r
\]

\[
r \approx 0.037
\]

16. (Compound Interest) How long, to the nearest whole month, will it take $40,000 to grow to $65,000 if 4\frac{1}{5}\% \text{ interest is compounded continuously}\? 

**Solution:** Substituting the values into the equation \( F = Pe^{rt} \) yields

\[
65000 = 40000e^{0.042t}
\]

\[
\ln \left( \frac{65000}{40000} \right) = \ln e^{0.042t}
\]

\[
\ln \left( \frac{65000}{40000} \right) = 0.042t
\]

\[
11.56 = t
\]

\[
11 + 0.5597 \cdot 12 = t
\]

11 years 7 months \( \approx t \)

17. (Compound Interest) How long, to the nearest whole month, will it take $40,000 to grow to $65,000 if 4\frac{1}{5}\% \text{ interest is compounded monthly}\? 

**Solution:** Substituting the values into the equation \( F = P \left( 1 + \frac{r}{n} \right)^n \) yields
18. (Compound Interest) Two certificates of deposit have the same effective annual yield. The first pays a rate of \( r \), compounded monthly. The second pays 6.03\%, compounded continuously. What is the value of \( r \)?

**Solution:** Want \( r \) so that

\[
P \left(1 + \frac{r}{n}\right)^n = P(1 + y) = Pe^{0.0603}
\]

Canceling \( P \) from both sides gives

\[
\left(1 + \frac{r}{n}\right)^n = e^{0.0603}
\]

Since we compound monthly, \( n = 12 \).

The equation becomes

\[
\left(1 + \frac{r}{12}\right)^{12} = e^{0.0603}
\]

Taking the 12th root gives

\[
1 + \frac{r}{12} = e^{0.0603}
\]

Solving for \( r \) yields

\[
r = 12 \left( e^{0.0603} - 1 \right)
\]

19. (Compound Interest) $5000 is invested in an account paying 5.5\%, compounded quarterly. How long will it take for the value of the account to reach $6000? Round your answer to the nearest hundredth of a year.

**Solution:** Given: \( P = 5000 \), \( F = 6000 \), \( r = .055 \), and \( n = 4 \). Solve for \( t \).

\[
F = P \left(1 + \frac{r}{n}\right)^{nt} \quad \rightarrow \quad \frac{F}{P} = \left(1 + \frac{r}{n}\right)^{nt}
\]

\[
\ln() \ln \left( \frac{F}{P} \right) = \ln \left(1 + \frac{r}{n}\right)^{nt} = nt \ln \left(1 + \frac{r}{n}\right)
\]
Solving for $t$ gives the general formula for time

$$t = \frac{\ln \left( \frac{F}{P} \right)}{n \ln \left( 1 + \frac{r}{n} \right)}$$

For our problem

$$t = \frac{\ln \left( \frac{6000}{5000} \right)}{4 \ln \left( 1 + \frac{.055}{4} \right)} \approx 3.34 \text{ years}$$

20. (Compound Interest) $6500$ is invested in an account paying 5.15%, compounded continuously. How long will it take for the value of the account to reach $10,000$? Round your answer to the nearest hundredth of a year.

**Solution:** Given: Compounded continuously, $P = 6500$, $F = 10,000$, $r = .0515$. Find $t$.

Recall that $F = Pe^{rt}$. Solve for $t$:

$$e^{rt} = \frac{F}{P} \quad (\text{divide by } P)$$

Take the natural log of both sides of the equation

$$\ln[e^{rt}] = \ln \left( \frac{F}{P} \right)$$

Using the basic property of logarithms $\ln(x^a) = a \ln x$

$$\ln[e^{rt}] = rt = \ln \left( \frac{F}{P} \right)$$

Dividing by $r$ yields the formula for time

$$t = \frac{1}{r} \ln \left( \frac{F}{P} \right)$$

For our problem, we want

$$t = \frac{1}{.0515} \ln \left( \frac{10000}{6500} \right) = 8.36 \text{ years.}$$
21. **(Simple Interest)** A total of $20,000 is invested in two funds paying 8% and 10% simple interest. If the interest for the first year is $2,000, how much of the $20,000 is invested at 10%?

**Solution:** Let \( I_1 \) be the interest earned on the investment at \( r_1 = .08 \), \( I_2 \) be the interest earned on the investment at \( r_2 = .10 \), and \( I_1 + I_2 = 2000 \) be the total interest from both investments.

Given: time \( t = 1 \) year and \( P = 20,000 \), we want to find \( x \), the amount invested at \( r_1 \).

Using Interest = Principal times rate times time with \( t = 1 \) we arrive at the equation \( I = x r_1 + (P - x)r_2 \). Substituting for \( r_1 \), \( r_2 \), \( I \), and \( P \) we have an equation for \( x \):

\[
2000 = .08x + .10(20000 - x)
\]

\[
= .1(20000) - .02x
\]

\[
= 2000 - .02x
\]

Simplifying yields \(.02x = 0 \Rightarrow x = 0\). This answer could have been deduced by observation.

22. **(Simple Interest)** A total of \( P \) dollars is invested in two funds paying \( r_1 \) and \( r_2 \) simple interest. If the interest for the first year is \( I \), how much of the initial investment \( P \) is invested at \( r_1 \)? Warning: your answer should be a formula involving \( r_1 \), \( r_2 \), \( P \), and \( I \).

**Solution:** Let \( I_1 \) be the interest earned on the investment at \( r_1 \), \( I_2 \) be the interest earned on the investment at \( r_2 \), and \( I_1 + I_2 = I \) be the total interest from both investments. Given: time \( t = 1 \) year and the initial investment is \( P \), we want to find \( x \), the amount invested at \( r_1 \).

Using Interest = Principal times rate times time with \( t = 1 \) we arrive at the equation \( I = I_1 + I_2 = x r_1 + (P - x)r_2 \). Solving this equation for \( x \) yields:

\[
I = x r_1 + (P - x)r_2 = P \cdot r_2 + (r_1 - r_2)x
\]

Solving for \( x \) yields:

\[
\begin{align*}
\frac{-P \cdot r_2}{r_1 - r_2} \rightarrow (r_1 - r_2)x & = I - P \cdot r_2 \\
\frac{-P \cdot r_2}{r_1 - r_2} \rightarrow x & = \frac{I - P \cdot r_2}{r_1 - r_2}
\end{align*}
\]

23. **(Simple Interest)** A total of $11,000 is invested in two funds paying 10% and 9% simple interest. If the interest for the first year is $1,000, how much of the $11,000 is invested at 10%?

**Solution:** Using the formula derived in problem 22 we get

\[
x = \frac{I - P \cdot r_2}{r_1 - r_2} = \frac{1000 - 11000(0.09)}{1.0 - 0.9} = \frac{1000 - 990}{0.1} = \frac{10}{0.1} = 1,000
\]
24. (Simple Interest) A total of $11,000 is invested in two funds paying 10% and 9% simple interest. If the interest for the first year is $1,000, how much of the $11,000 is invested at 10%? Solve the problem from scratch without using the formula derived in problem 22.

**Solution:** Let $I_1$ be the interest earned on the investment at $r_1 = .10$, $I_2$ be the interest earned on the investment at $r_2 = .09$, and $I_1 + I_2 = 1000$ be the total interest from both investments.

*Given:* time $t = 1$ year and $P = $11,000, we want to find $x$, the amount invested at $r_1$.

Using Interest = Principal times rate times time with $t = 1$ we arrive at the equation $I = xr_1 + (P - x)r_2$. Substituting for $r_1$, $r_2$, $I$, and $P$ we have an equation for $x$:

\[
1000 = .10x + .09(11000 - x)
\]

\[
= .09(11000) -.01x
\]

\[
.09(11000) -.01x
\]

\[
990 -.01x
\]

Simplifying yields

\[
.01x = 10 \implies x = 1000.
\]

25. (Simple Interest) A total of $20,000 is invested in two funds paying 8% and 10% simple interest. If the interest for the first year is $2,000, how much of the $20,000 is invested at 10%?

**Solution:** Let $I_1$ be the interest earned on the investment at $r_1 = .09$, $I_2$ be the interest earned on the investment at $r_2 = .11$, and $I_1 + I_2 = 1180$ be the total interest from both investments.

*Given:* time $t = 1$ year and $P = $12,000, we want to find $x$, the amount invested at $r_1$.

Using Interest = Principal times rate times time with $t = 1$ we arrive at the equation $I = xr_1 + (P - x)r_2$. Substituting for $r_1$, $r_2$, $I$, and $P$ we have an equation for $x$:

\[
1180 = .09x + .11(12000 - x)
\]

\[
= .11(12000) -.02x
\]

\[
.1(12000) +.01(12000) -.02x
\]

\[
1200 + 120 -.02x
\]

Simplifying yields

\[
.02x = 140 \implies x = 7000.
\]