Mathematics for Business Decisions, Part I

Problem Set 9: Finite Random Variables, Binomial Distributions, and Random Samples

Solutions

NOTE: For more practice problems with solutions, see my class notes handout.

Elementary-Level Problems

All problems are Finite-Random-Variable problems

Recall the following definitions: \( f_X(x) = P(X = x) \) and \( F_X(x) = P(X \leq x) \).

1-6. A finite random variable \( X \) has the following values and probabilities.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X=x) )</td>
<td>0.2</td>
<td>0.1</td>
<td>0.3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(1) The p.m.f., \( f_X \) has \( f_X(0) = .1 \)  
(2) The c.d.f., \( F_X \) has \( F_X(0) = .3 \)  
(3) The p.m.f., \( f_X \) has \( f_X(2) = 0 \)  
(4) The c.d.f., \( F_X \) has \( F_X(2) = .6 \)  
(5) \( P(X = 0) = f_X(0) = .1 \)  
(6) \( P(X \leq 0) = F_X(0) = .3 \)

7-12 A finite random variable \( X \) has the following values and probabilities.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X=x) )</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(7) The p.m.f., \( f_X \) has \( f_X(1) = .1 \)  
(8) The p.m.f., \( f_X \) has \( f_X(10) = 0 \)  
(9) The c.d.f., \( F_X \) has \( F_X(1) = .1 \)  
(10) The c.d.f., \( F_X \) has \( F_X(10) = 1 \)  
(11) \( P(X = 2) = .2 \)  
(12) \( P(X \leq 2) = .3 \)

13-18 A finite random variable \( X \) has the following values and probabilities.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-1</th>
<th>0</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X=x) )</td>
<td>0.3</td>
<td>0.1</td>
<td>0.2</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(13) The p.m.f., \( f_X \) has \( f_X(4) = 0 \)  
(14) The p.m.f., \( f_X \) has \( f_X(-5) = .3 \)  
(15) The c.d.f., \( F_X \) has \( F_X(4) = .6 \)  
(16) The c.d.f., \( F_X \) has \( F_X(-5) = .3 \)  
(17) \( P(X = 2) = 0 \)  
(18) \( P(X \leq 0) = .6 \)
19. A finite random variable $X$ has the following cumulative distribution $F_X(x)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_X(x)$</td>
<td>0.3</td>
<td>0.5</td>
<td>0.8</td>
<td>1</td>
</tr>
</tbody>
</table>

Graph $F_X(x)$. 
20. A finite random variable $X$ has the following cumulative distribution $F_X(x)$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_X(x)$</td>
<td>0.1</td>
<td>0.3</td>
<td>0.6</td>
<td>1</td>
</tr>
</tbody>
</table>

Graph $F_X(x)$. 
21-23 Consider the graph of the cumulative distribution function \( y = F_X(x) \).

21. Write down the formula for \( F_X(x) \) as a piecewise step function.

\[
F_X(x) = \begin{cases} 
0 & \text{if } x < 0 \\
.3 & \text{if } 0 \leq x < 1 \\
.5 & \text{if } 1 \leq x < 2 \\
.8 & \text{if } 2 \leq x < 3 \\
1 & \text{if } 3 \leq x 
\end{cases}
\]

**Solution:**

22. Use the above graph to fill in the information in the table below.

**Solution:**

<table>
<thead>
<tr>
<th>Range of ( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_X(x) = P(X = x) )</td>
<td>.3</td>
<td>.2</td>
<td>.3</td>
<td>.2</td>
</tr>
<tr>
<td>( F_X(x) = P(X \leq x) )</td>
<td>.3</td>
<td>.5</td>
<td>.8</td>
<td>1</td>
</tr>
</tbody>
</table>

**Warning:** You may not need to use all of the columns to complete the table.

23. Compute \( F_X(2.5) \).

**Solution:** \( F_X(2.5) = .8 \). We can see the answer directly from the graph above.
24-25. Let $X$ be a finite random variable with the following cumulative probability distribution:

$$F_X(x) = \begin{cases} 
0 & \text{if } x < -3 \\
.4 & \text{if } -3 \leq x < -1 \\
.7 & \text{if } -1 \leq x < 1 \\
1 & \text{if } 1 \leq x
\end{cases}$$

24. What is the range of $X$?

**Solution:** Range of $X = \{-3, -1, 1\}$.

25. Find all of the values for $f_X(x)$.

**Solution:** $f_X(-3) = .4$; $f_X(-1) = .3$; $f_X(1) = .3$. Note: the sum of the probabilities is 1.

**Applications: Binomial distributions**

**Problems 26-30 are Binomial Distribution problems.** You can solve these problems using the Excel function `BINOMDIST` found under the insert function in Excel. Recall: The arguments of the function `BINOMDIST` are:

- $x =$ Number_of_successes in the trials,
- $n =$ Trials = the number of independent trials,
- $\theta =$ Probability_success = the probability of success on each trail,
- Cumulative = a logical value: for the cumulative distribution function, use TRUE; for the probability mass function, use FALSE.

Thus, we can express the Binomial p.m.f. and c.d.f. as

$$\begin{align*}
f_X(x) &= \text{Binomdist}(x, n, \theta, FALSE) \\
F_X(x) &= \text{Binomdist}(x, n, \theta, TRUE)
\end{align*}$$

Use this information to solve the following problems.

26. Find the probability of getting exactly 6 heads in 12 flips of a fair coin.

**Solution:**

$$\begin{align*}
x &= 6 \\
\text{Let } n &= 12. \text{ Then } \\
\theta &= .5
\end{align*}$$

$$P(X = 6) = f_X(6) = \text{Binomdist}(6, 12, .5, False) = 0.225586$$
27. Find the probability of getting exactly 2 heads in 3 flips of a fair coin.

**Solution:**

\[
\begin{align*}
x &= 2 \\
n &= 3 \\
\theta &= 0.5
\end{align*}
\]

\[P(X = 2) = f_X(2) = \text{Binomdist}(2, 3, 0.5, \text{False}) = 0.375\]

28. Find the probability of getting at most 2 heads in 3 flips of a fair coin.

**Solution:**

\[
\begin{align*}
x &= 2 \\
n &= 3 \\
\theta &= 0.5
\end{align*}
\]

\[P(X \leq 2) = F_X(2) = \text{Binomdist}(2, 3, 0.5, \text{True}) = 0.875\]

29. Find the probability of getting at least 3 heads in 5 flips of a fair coin.

**Solution:**

\[
\begin{align*}
x &= 3 \\
n &= 5 \\
\theta &= 0.5
\end{align*}
\]

\[P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - \text{Binomdist}(2, 5, 0.5, \text{True}) = 0.5\]

30. Roll a die 18 times. On each roll, define a success as the event that the face value is one. Find the probability of getting at least 3 ones in 18 rolls of the die.

**Solution:**

\[
\begin{align*}
x &= 3 \\
n &= 18 \\
\theta &= 1/6
\end{align*}
\]

\[P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - \text{Binomdist}(2, 18, 1/6, \text{True}) = 0.597346\]
Random Sampling (31-32)
Let $X$ be the score on the last exam of a randomly selected student in your Business Mathematics I class. Suppose that only five people in your class agree to tell their grades: 50, 55, 60, 65, 70. Use this information to estimate

31. $E(X)$

(a) 0  (c) 20  (e) 40  (g) 60  (i) 80  (k) 100
(b) 10  (d) 30  (f) 50  (h) 70  (j) 90  (l) none of the these

Solution: With a small sample set, the best that we can do is to use the average to approximate the expected value of $X$.

$$E(X) \approx \bar{X} = \frac{50 + 55 + 60 + 65 + 70}{5} = 60$$

and 32. $F_X(60)$.

(a) 0  (c) .2  (e) .4  (g) .6  (i) .8  (k) 1
(b) .1  (d) .3  (f) .5  (h) .7  (j) .9  (l) none of the these

Solution:

$$F_X(60) = P(X \leq 60) = \frac{\# \text{ of scores } \leq 60}{\# \text{ of scores in sample}} = \frac{\#(50, 55, 60)}{5} = \frac{3}{5} = .6.$$