Business Mathematics I: Math 173

Sample Exam 1 - Solutions

Name:

Print your name neatly. If you forget to write your name, or write so sloppy that I can’t read it, you can lose all of the points! Answer all the questions that you can. Circle your answer.

All problems are worth 1 point each.

Do any 25 of the 26 problems. You may do all 26 if you like, but I will stop grading after you have 25 problems correct. In other words, you cannot score more than 100% on this exam. You must show how you arrived at the answer in order to receive any credit for the problem. Problems for which only an answer is given with no supporting work, will receive a score of zero for that problem.

STUDY GUIDE FOR EXAM 1:

Topics covered on this exam: Simple and Compound interest, Evaluating Functions in Excel, Summation, Venn Diagrams.

What homework sets should you review for this exam: Homework sets 1, 2, 3, 4.

Warning: The actual exam will need not be the same questions, but the number of problems, types of problems, and level of difficulty will be similar.
1. **(Compound Interest)** You have $10,000 to invest. You are considering two investment schemes, each offering an annual interest rate of 5%. Scheme A offers 5% interest compounded continuously, while scheme B offers 5% interest compounded quarterly. Which investment scheme will result in the largest future value of your principal?

**Solution:** Choose scheme A. Recall: if $P$, $r$, and $t$ are fixed, then your investment will grow the fastest under continuous compound interest. This scheme will give you the largest future value.

2. **(Compound Interest)** Below are two graphs of the future value of the principal $P = $1000 as a function of time. The interest rate for both graphs is $r = .04$. One graph grows at a simple interest rate, while the other grows at a compound interest rate with $n = 12$. Both graphs are plotted over the time interval starting at time $t = 0$ and ending at time $t = 30$ years. Clearly label which one is the graph of simple interest growth. Warning: If I can’t tell which graph you indicate you will receive a zero for the problem.

![Graph of compound interest vs. simple interest](image)

**Solution:** Recall: An investment of $P$ dollars under a simple interest rate $r$ grows linearly in time. Hence, the graph of simple interest is linear. Moreover, for fixed $P$ and $r$, it is the slowest-growing interest scheme in time. The lower linear graph is the simple interest graph.
3. **(Compound Interest: Future Value)** What is the value of $100 after 10 years at 5% interest, compounded monthly? Round to the nearest dollar.

(a) 1056  (c) 569  (e) 730  (g) 297  (i) 953  (k) 9394  
(b) 723  (d) 165  (f) 704  (h) 367  (j) 1  (l) none of the these

**Solution:** Given: $P = 100$, $t = 10$, $r = .05$, and $n = 12$. Want: $F$.

The discrete interest formula is $F = P \left(1 + \frac{r}{n}\right)^{nt}$. Upon substituting the given values into the formula we arrive at $F = 100 \left(1 + \frac{.05}{12}\right)^{12(10)} = 164.7 \approx 165$.

4. **(Compound Interest: yield)** What is the effective annual yield $y$ for problem 3? (Use 3 decimal place accuracy).

(a) .307  (c) .517  (e) .023  (g) .051  (i) .027  (k) .039  
(b) .024  (d) .039  (f) .105  (h) .002  (j) .095  (l) none of the these

**Solution:** The formula for effective annual yield in the case of discrete compounding is $y = \left(1 + \frac{r}{n}\right)^n - 1$. Substituting the values from problem 3 yields $y = \left(1 + \frac{.05}{12}\right)^{12} - 1 \approx .051$.

5. **(Compound Interest: Future Value)** Find the value of $1,250,000 after 12 years and 6 months, if it is invested at a rate of $\frac{5.1}{8}\%$, compounded continuously.

**Solution:** Given: $P = 1,250,000$, $r = (41/(8*100)) = 0.05125$, $t = 12.5$ years.

$$F = Pe^{rt}$$

$$= 1250000e^{(0.05125\times12.5)}$$

$$= 2,372,083.19$$
6. **(Compound Interest: time to double investment)** You want to invest P dollars. How long will it take to double your investment at an annual interest rate of 10%, compounded continuously? Round your answer to the nearest year.

(a) 1  (c) 3  (e) 5  (g) 7  (i) 9  (k) 11  
(b) 2  (d) 4  (f) 6  (h) 8  (j) 10  (l) none of these

**Solution:** Given: \( F = 2P \), \( r = .10 \), and compounding scheme is continuous. Want: \( t_{\text{double}} \).

Substituting these values into the compound interest formula \( F = Pe^{rt} \) yields

\[
2P = Pe^{rt} \quad \Rightarrow \quad 2 = e^{rt} \quad (\text{divide by } P)
\]

\[
\Rightarrow \quad \ln(2) = rt \quad (\text{Take the natural log of both sides})
\]

\[
\Rightarrow \quad t = \frac{\ln(2)}{r} \approx \frac{.69}{.1} = 6.9 \quad (\text{Solve for } t)
\]

7. **(Compound Interest: time to double investment)** You want to invest P dollars. How long will it take to double your original investment at an annual interest rate of 4%, compounded quarterly? (round your answer to the nearest year).

(b) 10  (c) 12  (e) 14  (g) 16  (i) 18  (k) 20  
(b) 11  (d) 13  (f) 15  (h) 17  (j) 19  (l) none of the these

**Solution:** Given: \( F = 2P \), \( r = .069 \), and \( n = 4 \). Want: \( t \).

Substituting these values into the compound interest formula \( F = Pe^{rt} \) yields

\[
2P = P \left(1 + \frac{r}{n}\right)^{nt} \quad \Rightarrow \quad 2 = \left(1 + \frac{r}{n}\right)^{nt} \quad (\text{divide by } P)
\]

\[
\Rightarrow \quad \ln(2) = nt \ln\left(1 + \frac{r}{n}\right) \quad (\text{Take the natural log of both sides})
\]

\[
\Rightarrow \quad t = \frac{\ln(2)}{n \ln\left(1 + \frac{r}{n}\right)} \approx \frac{\ln(2)}{.04} = 17 \quad (\text{Solve for } t)
\]
8. *(Compound Interest)* What annual rate $r$, compounded continuously, would have the same yield as an annual rate of 6%, compounded weekly? Round off your answer to the nearest hundredth of a percent (2 decimal place accuracy).

(a) .02    (c) .04    (e) .06    (g) .08    (i) .10    (k) .12
(b) .03    (d) .05    (f) .07    (h) .09    (j) .11    (l) none of the these

**Solution:** Given: $r = .06$, $n = 52$ (discrete case). Want: $r$ for the compound interest scheme.

The relevant formulas for yield are

\[
\begin{align*}
\text{Continuous compounding case: } & y = e^r - 1 \\
\text{Discrete compounding case: } & y = \left(1 + \frac{r}{n}\right)^n - 1
\end{align*}
\]

Equating the two expressions for yield gives

\[
e^r - 1 = \left(1 + \frac{.06}{52}\right)^{52} - 1 \quad \Rightarrow \quad e^r = \left(1 + \frac{.06}{52}\right)^{52}
\]

\[
\Rightarrow \quad r = \ln\left(1 + \frac{.06}{52}\right)^{52} = 52 \ln\left(1 + \frac{.06}{52}\right) \approx 52 \left(\frac{.06}{52}\right) = .06
\]

**Fast way:** Recall: $y \approx r$ for both the discrete and continuous case provided that $r < 0.2$.

9. *(Compound Interest)* How long, to the nearest whole month, will it take $40,000$ to grow to $65,000$ if $4\frac{1}{5}\%$ interest is compounded continuously?

**Solution:** Substituting the values into the equation $F = Pe^{rt}$ yields

\[
\frac{65000}{40000} = e^{0.042t}
\]

\[
\ln\left(\frac{65000}{40000}\right) = \ln e^{0.042t}
\]

\[
\ln\left(\frac{65000}{40000}\right) = 0.042t
\]

\[
11.56 = t
\]

\[
11 + 0.5597 \cdot 12 = t
\]

11 years 7 months $\cong t$
10. **(Compound Interest)** Two certificates of deposit have the same effective annual yield. The first pays a rate of \( r \), compounded monthly. The second pays 6.03\%, compounded continuously. What is the value of \( r \) ?

**Solution:** Want \( r \) so that

\[
P \left(1 + \frac{r}{n}\right)^n = Pe^{0.0603}
\]

Canceling \( P \) from both sides gives

\[
\left(1 + \frac{r}{n}\right)^n = e^{0.0603}
\]

Since we compound monthly, \( n = 12 \). The equation becomes

\[
\left(1 + \frac{r}{12}\right)^{12} = e^{0.0603}
\]

Taking the 12th root gives

\[
1 + \frac{r}{12} = e^{\frac{0.0603}{12}}
\]

Solving for \( r \) yields

\[
r = 12 \left(e^{\frac{0.0603}{12}} - 1\right)
\]

11. **(Simple Interest)** A total of $12,000 is invested in two funds paying 9\% and 11\% simple interest. If the interest for the first year is $1,180, how much of the $12,000 is invested at 9\%?

**Solution:** Let \( I_1 \) be the interest earned on the investment at \( r_1 = 0.09 \), \( I_2 \) be the interest earned on the investment at \( r_2 = 0.11 \), and \( I_1 + I_2 = 1180 \) be the total interest from both investments.

*Given:* time \( t = 1 \) year and \( P = $12,000 \), we want to find \( x \), the amount invested at \( r_1 \).

Using Interest = Principal times rate times time with \( t = 1 \) we arrive at the equation

\[
I = x r_1 + (P - x) r_2.
\]

Substituting for \( r_1 \), \( r_2 \), \( I \), and \( P \) we have an equation for \( x \):

\[
1180 = 0.09x + 0.11(12000 - x)
\]

\[
= 0.11(12000) - 0.02x
\]

\[
= 0.11(12000) + 0.01(12000) - 0.2x
\]

\[
1200 + 120 - 0.2x
\]

Simplifying yields

\[
0.02x = 140 \quad \Rightarrow \quad x = 7000.
\]
For problems 12-15 convert the following mathematical expressions into Excel-function format. For example, the mathematical function \( f(x) = \frac{x+1}{x^2-x+1} \) when expressed in Excel-function format becomes \( f(x) = (x+1)/(x^2-x+1) \). The parentheses around the denominator are necessary. The expression \( f(x) = (x+1)/x^2-x+1 \) when written in mathematical notation is \( f(x) = \frac{x+1}{x^2-x+1} \). This is a very different result from what was intended.

12. (Excel-Function notation): Write \( f(x) = e^{-x^2} \) in Excel-function format.

Solution: \( f(x) = \text{Exp}(-(x^2)) \)

Warning: Excel interprets \(-x^2\) as \((x^2)\), so we must write \(-x^2\) as \(-(x^2)\). This is a work-around for a defect in the Excel programming language. Live with it!

13. (Excel-Function notation): Write \( F = P(1+r/n)^n \) in Excel-function format.

Solution: \( F = P*(1+r/n)^n \)

14. (Excel-Function notation): Write \( F = Pe^{rt} \) in Excel-function format.

Solution: \( F = P*\text{Exp}(r*t) \)

15. (Excel-Function notation): Write \( f(x) = \frac{x^2 + 2x + 1}{1 + x^2} \) in Excel-function format.

Solution: \( (x^2 + 2x + 1)/(1+x^2) \)

NOTE: We needed to put both numerator and denominator in parentheses.

Next, we try going in reverse. For problems 16-20 convert the following equations written in Excel-function format into mathematical expressions. For example, the function \( f(x) = (x^2 - 7x + 1)/(x^2 - 3x + 19) \) is written in Excel-function format. It becomes \( f(x) = \frac{x^2 - 7x + 1}{x^2 - 3x + 19} \) when it is expressed in standard mathematical notation.

16. (Excel-Function notation): \( f(x) = x^2 - 3x + 5 \)

Solution: \( f(x) = x^2 - 3x + 5 \)

17. (Excel-Function notation): \( y = (1+r/n)^n \) \( - 1 \)

Solution: \( y = \left( 1 + \frac{r}{n} \right)^n - 1 \)
18. (Excel-Function notation): \( f(x) = 5x^2 e^x \)

**Solution:** \( f(x) = 5x^2 e^x \)

19. (Summation) Express the given sum using summation notation.

\[
\frac{1}{1^2 + 1} + \frac{1}{2^2 + 1} + \frac{1}{3^2 + 1} + \cdots + \frac{1}{100^2 + 1} =
\]

**Solution:**
\[
\frac{1}{1^2 + 1} + \frac{1}{2^2 + 1} + \frac{1}{3^2 + 1} + \cdots + \frac{1}{100^2 + 1} = \sum_{i=1}^{100} \frac{1}{i^2 + 1}
\]

20. (Summation) Expand the following expression written in summation notation. 
*Do not attempt to evaluate the sum!*

\[
\sum_{i=1}^{5} x_i^2 f(x_i) =
\]

**Solution:**
\[
\sum_{i=1}^{5} x_i^2 f(x_i) = x_1^2 f(x_1) + x_2^2 f(x_2) + x_3^2 f(x_3) + x_4^2 f(x_4) + x_5^2 f(x_5)
\]

21. (Summation) Evaluate the sum

\[
\sum_{n=-3}^{2} \frac{1}{i^2 + 1} . \text{ Here } a_i = a(i) = \frac{1}{i^2 + 1}. \text{ For sum: start value = -3, stop value = 2}
\]

**Solution:**
\[
\sum_{n=-3}^{2} \frac{1}{i^2 + 1} = \frac{1}{(-3)^2 + 1} + \frac{1}{(-2)^2 + 1} + \frac{1}{(-1)^2 + 1} + \frac{1}{0^2 + 1} + \frac{1}{1^2 + 1} + \frac{1}{2^2 + 1}
\]
\[
= \frac{1}{10} + \frac{1}{5} + \frac{1}{2} + \frac{1}{5} + \frac{1}{2} + \frac{1}{5} = 2 + \frac{2}{5} + \frac{1}{10} = 2 + \frac{1}{2} = \frac{5}{2}
\]

22. (Summation) Expand the following expression written in summation notation. 
*Do not attempt to evaluate the sum!*

\[
\sum_{i=1}^{5} x_i P(X = x_i) =
\]

**Solution:**
\[
\sum_{i=1}^{5} x_i P(X = x_i) = x_1 P(X = x_1) + x_2 P(X = x_2) + x_3 P(X = x_3) + x_4 P(X = x_4) + x_5 P(X = x_5)
\]
For **problems 23-26** shade in the appropriate regions for the given diagram

23.

\[ A \cup B \]

24.

\[ B - A = A^c \cap B \]

25.

\[ A \cap B \cap C \]

26.

\[ (A \cup B) \cap C \]