

Algebra Primer  
A quick overview of basic algebraic concepts  
(for physics students everywhere!)

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# Contents

<b>1</b>	<b>A Review of Elementary Algebra</b>	<b>2</b>
1.1	Mathematical terminology: expressions, equations, terms, and factors . . .	2
1.2	The order of operations . . . . .	4
1.3	Some examples with order of operations . . . . .	5
1.4	Solving linear equations . . . . .	8
1.4.1	Solving an equation: algebraic method: . . . . .	8
<b>2</b>	<b>A Review of College Algebra</b>	<b>9</b>
2.1	Properties of real numbers and the real number line . . . . .	9
2.2	Properties of the absolute-value function . . . . .	10
2.3	Properties of exponentials . . . . .	10
2.4	A review of basic properties of polynomials . . . . .	12
2.4.1	Completing the square . . . . .	12
2.4.2	Completing the square to graph a parabola . . . . .	15
2.4.3	Geometric proof for the pythagorean theorem . . . . .	16
2.4.4	Factoring Trinomials . . . . .	16
2.4.5	Special products and factoring formulas . . . . .	17
2.4.6	Practice problems with multiplication of polynomials . . . . .	20
2.5	Properties of complex numbers . . . . .	22
2.6	Working with inequalities . . . . .	22
2.7	Distance, midpoints, and intercepts . . . . .	23
2.8	Variation/Proportional-to . . . . .	24
2.9	A review of lines . . . . .	24
2.10	A review of the properties of graphs and functions . . . . .	25
2.10.1	Basic properties of functions . . . . .	26
2.11	Exponentials and Logarithms . . . . .	34
2.12	Rules for systems of equations . . . . .	36

# 1 A Review of Elementary Algebra

## 1.1 Mathematical terminology: expressions, equations, terms, and factors

In english the term expression is often use to represent a statement, or the formulation of an idea. In mathematics the meaning is similar the this, only it represents a complete or partial mathematical idea.

**Informal definition 1.** A *mathematical expression*, or expression for short, is a combination of mathematical symbols put together to convey a mathematical idea. It may contain mathematical grouping symbols and even english punctuation, and other syntactic symbols. An expression acts as a mathematically statement, or “mathematical sentence”. It should be logically well-formed. Here the word symbols refers to numbers, parameters, variables, operations, functions, and other mathematical symbols.

Mathematical expressions vary wildly in form here are some examples:

### Example 1. Examples of expressions

- $ax^2 + bx + c$
- $\frac{\sqrt{2y}}{g}$
- $a^2 + 2ab + b^2$
- $(a + b)^2$
- $\frac{3x + 1}{x^2 - 3x + 5}$

**Informal definition 2.** A *mathematical equation*, or equation for short, is a formula of the form  $A = B$ , where  $A$  and  $B$  are expressions containing one or several variables called unknowns, and the equal sign “=” denotes the a logical binary relation symbol<sup>1</sup>.

### Example 2. Examples of equations

- $y = ax^2 + bx + c$
- $y = \frac{\sqrt{2y}}{g}$
- $(a + b)^2 = a^2 + 2ab + b^2$

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<sup>1</sup>If you want more detail, seek a good axiomatic mathematical logic book!

**Comments:**

- Although an equation of the form  $A = B$  looks as though it is written in the form of proposition, it is not. It is important to understand that an equation is not a statement, but rather a problem consisting in finding unknown values of parameters/variables that appear in the equation. When the unknowns are determined by various mathematical methods and substituted to the unknowns, the result yields equal values of expressions A and B.
- The main difference between a mathematical expression and an equation is that an expression does not need to contain an equal sign, nor is it designed to be “solved” for any unknowns in it; whereas, an equation must contain an equal sign and is a mathematical puzzle that must be solved.

**Informal definition 3.** A *term* in an equation is like a building-block, or component, of an equation. It is the mathematical analogy to a noun phrase in a sentence in the study of languages.

**Example 3. Examples of terms in expressions and equations**

- In the expression  $ax^2 + bx + c$  the terms are:  $ax^2$ ,  $bx$ , and  $c$ .
- In the equation  $(a + b)^2 = a^2 + 2ab + b^2$  a term is  $a^2$ .

**Informal definition 4.** A *factor* in a mathematical equation is simply an expression that is multiply another expression.

**Example 4. Examples of factors in expressions and equations**

- In the expression  $ax^2 + bx + c$  the term  $bx$  is composed of two factors:  $b$  and  $x$ .
- In the expression  $f(x) \left(1 + \frac{2}{x}\right)$  the  $f(x)$  is a factor multiplying the expression  $\left(1 + \frac{2}{x}\right)$ .
- In the expression  $\frac{\sqrt{2y}}{g}$ , the numerator and the denominator of the fraction are each factors of the factor (expression) as a whole.

## 1.2 The order of operations

### Grouping Symbols:

The most common grouping symbols are

(   ), [   ], {   }, and the fraction bar —

Grouping symbols help determine the order in which mathematical expressions are evaluated.

**Algorithm for working with Grouping Symbols:** If the expression contains grouping symbols, do all the calculations within each pair of grouping symbols, working from the innermost pair to the outermost pair. Starting with the innermost pair, and working left to right, evaluate the expression in the following hierarchical order:

Step 1. Evaluate all exponents

Step 2. Evaluate all  $\times$ ,  $\div$

Step 3. Evaluate all  $+$ ,  $-$

**NOTE:** There are no grouping symbols in the hierarchy because, by assumption, you are already working in the innermost pair of grouping symbols.

After evaluating the expression you can remove the innermost grouping symbols. Repeat the process until you have completely evaluated the expression.

**Order of Operations Hierarchy:** When evaluating an expression, always work from left to right.

Step 1. (   ), [   ], {   }, and the fraction bar —

Step 2. Exponents

Step 3.  $\times$ ,  $\div$

Step 4.  $+$ ,  $-$

**Warning:** Excel follows this hierarchy!

**Example:** Write out how you would evaluate the following expression in Excel:

$$\frac{2}{n(n+1)},$$

where the value of  $n$  is found in the cell A2.

**Answer:** In Excel, you would write:  $2/(A2(A2+1))$ .

### 1.3 Some examples with order of operations

**Example 5.**  $3 \cdot 2^3 - 4 = 3 \cdot 8 - 4 = 24 - 4 = 20$

**Example 6.**  $30 - 4 \cdot 5 + 9 = 30 - 20 + 9 = 19$

**Example 7.** Working with grouping symbols:

$$\begin{aligned} 5 - 6 + 4 &= -1 + 4 = 3 \\ 5 - (6 + 4) &= 5 - 10 = -5 \end{aligned}$$

**Example 8.**  $(6 - 3)^2 = 3^2 = 9$

**Example 9.**  $8^2 + 2(10 - 4 \cdot 2)$

$$\begin{aligned} 8^2 + 2(10 - 4 \cdot 2) &= 8^2 + 2(10 - 8) \\ &= 8^2 + 2(2) \\ &= 64 + 4 \\ &= 68 \end{aligned}$$

Now try  $7^2 - 5(8 - 3 \cdot 2)$

$$\begin{aligned} 7^2 - 5(8 - 3 \cdot 2) &= 7^2 - 5(8 - 6) \\ &= 7^2 - 5(2) \\ &= 49 - 10 \\ &= 39 \end{aligned}$$

**Example 10.**  $\frac{3(15) - 12}{2 + 3^2}$

Simplify the top and bottom first:

$$\begin{aligned} \frac{3(15) - 12}{2 + 3^2} &= \frac{45 - 12}{2 + 9} \\ &= \frac{33}{11} \\ &= 3 \end{aligned}$$

**Example 11.**  $a + b \{c - (d + e)^2 [fg - h(i + j)]\} = k$

Solve for  $f$  if  $a = 12$ ,  $b = 3$ ,  $c = 5$ ,  $d = 1$ ,  $e = 2$ ,  $g = 3$ ,  $h = 0$ ,  $i = 7$ ,  $j = \pi$ , and  $k = 9$ .

We substitute all the given numbers into the equation and simplify, using order of operations.

$$12 + 3 \{5 - (1 + 2)^2 [f \cdot 3 - 0(7 - \pi)]\} = 9$$

(We can't simplify  $7 - \pi^\pi$ , but it's multiplied by zero)

$$\begin{aligned} \xrightarrow{\text{simplify inside } [ ]} \quad & 12 + 3\{5 - 3^2[3f]\} = 9 \\ & 12 + 3\{5 - 27f\} = 9 \end{aligned}$$

(We want to isolate the  $5 - 27f$ )

$$\begin{aligned} \xrightarrow{-12} \quad & 3\{5 - 27f\} = -3 \\ \xrightarrow{\div 3} \quad & 5 - 27f = -1 \\ \xrightarrow{-5} \quad & -27f = -1 - 5 = -6 \\ \xrightarrow{\div (-27)} \quad & f = \frac{6}{27} = \frac{2}{9} \end{aligned}$$

**Example 12.** Solve for  $x$ :  $a - 2[b - 3(c - x)] = 6$

$$\begin{aligned} \xrightarrow{-a} \quad & -2[b - 3c + 3x] = 6 - a \\ \xrightarrow{\div (-2)} \quad & b - 3c + 3x = \frac{6 - a}{-2} = \frac{a - 6}{2} \\ \xrightarrow{-(b - 3c)} \quad & 3x = \frac{a - 6}{2} - (b - 3c) \\ \xrightarrow{\div 3} \quad & x = \frac{a - 6}{6} - \frac{(b - 3c)}{3} = \frac{a - 6 - 2(b - 3c)}{6} = \frac{a - 2b + 6c - 6}{6} \\ & = \frac{a - 2b}{6} + c - 1 \end{aligned}$$

**Problem 1:** Solve for  $x$ :  $a^2x + (a - 1) = (a + 1)x$

**Problem 2:** Solve for  $\mathcal{P}(A \cap B)$  given  $\mathcal{P}(A) = 0.3$ ,  $\mathcal{P}(B) = 0.6$ , and  $\mathcal{P}(A \cup B) = 0.7$ .

**Example 9** If  $x = yz + c(9 - a)$  and  $x = 10$ ,  $c = 3$ ,  $y = \frac{1}{2}$ , and  $a = 3$ , then what is the value of  $z$ ?

We substitute the given values and solve:

$$\begin{array}{rcl}
 & & 10 = \frac{1}{2}z + 3(9 - 3) \\
 \xrightarrow{\text{simplify inside } ( )} & & 10 = \frac{1}{2}z + 3(6) \\
 \xrightarrow{\text{multiply}} & & 10 = \frac{z}{2} + 18 \\
 \xrightarrow{-18} & & -8 = \frac{z}{2} \\
 \xrightarrow{\cdot 2} & & -16 = z
 \end{array}$$

**Example 13.** Retail stores use the formula

$$R = M + C$$

where  $R$  is the retail selling price,  $M$  is the markup on an item, and  $C$  is the cost that the store paid for the item. If  $R = 25$  and  $C = 18$ , what is  $M$ ?

$$\begin{array}{rcl}
 & & 25 = M + 18 \\
 \xrightarrow{-18} & & 7 = M
 \end{array}$$

**Example 14.** If  $xy + yz + xz = 20$ , and if  $x = 10$  and  $z = 3$ , what is  $y$ ?

We substitute the given numbers and solve for  $y$ :

$$\begin{array}{rcl}
 & & 10y + 3y + 10 \cdot 3 = 20 \\
 \xrightarrow{\text{Combine like terms}} & & 13y + 30 = 20 \\
 \xrightarrow{-30} & & 13y = -10 \\
 \xrightarrow{\div 13} & & y = -\frac{10}{13}
 \end{array}$$



## 1.4 Solving linear equations

A linear equation in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where  $a$  and  $b$  are real numbers and  $x$  is the variable.

Linear equations can be disguised, so you should be ready for this.

**Example:** (converting a ratio into a linear equation)

Convert the given equation into a linear equation and solve for  $x$ .

$$\begin{aligned} & \frac{2x - 3}{5x + 4} = -2 \quad (\text{assume } x \neq -4/5) \\ \xrightarrow{\times(5x+4)} & 2x - 3 = -2(5x + 4) \\ \xrightarrow{\text{distribute}} & 2x - 3 = -10x - 8 \\ \xrightarrow{+10x} & 12x - 3 = -8 \\ \xrightarrow{+3} & 12x = -5 \\ \xrightarrow{\div 12} & x = -5/12 \end{aligned}$$

### 1.4.1 Solving an equation: algebraic method:

Use the rules of algebra to isolate the unknown  $x$  on one side of the equation. Example:

$$\begin{aligned} 2x &= 6 - x \\ 3x &= 6 && (\text{add } x) \\ x &= 2 && (\text{divide by } 3) \end{aligned}$$

## 2 A Review of College Algebra

### 2.1 Properties of real numbers and the real number line

**The fundamental properties of real numbers:**

$$\text{Commutative properties:} \quad a + b = b + a \quad ab = ba$$

$$\text{Associative properties:} \quad (a + b) + c = a + (b + c) \quad (ab)c = a(bc)$$

$$\text{Distributive properties:} \quad a(b + c) = ab + ac \quad (b + c)a = ab + ac$$

**Properties of negatives:**

$$(-1)a = -a$$

$$-(-a) = a$$

$$(-a)b = a(-b) = -(ab)$$

$$(-a)(-b) = ab$$

$$-(a + b) = -a - b$$

$$-(a - b) = b - a$$

**Properties of fractions:**

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{ac}{bc} = \frac{a}{b}$$

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc$$

**Intervals:**

$$(a, b) = \{x \mid a < x < b\}$$

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, \infty) = \{x \mid a < x\}$$

$$[a, \infty) = \{x \mid a \leq x\}$$

$$(-\infty, b) = \{x \mid x < b\}$$

$$(-\infty, b] = \{x \mid x \leq b\}$$

$$(-\infty, \infty) = \mathbb{R}: \text{ the set of all real numbers}$$

## 2.2 Properties of the absolute-value function

**Definition of absolute value:**

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

**Properties of absolute value:**

$$|a| \geq 0$$

$$|a| = |-a|$$

$$|ab| = |a||b|$$

$$\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$$

**Distance between points on the real line:**

$$d(a, b) = |b - a|$$

## 2.3 Properties of exponentials

**Exponential notation:**

If  $a$  is a real number and  $n$  is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

**Zero and negative exponents:**

If  $a$  is a real number and  $n$  is a positive integer, then

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}$$

**Laws of exponents:**

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left( \frac{a}{b} \right)^n = \frac{a^n}{b^n}$$

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

$$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$$

**Scientific notation:**

A positive number  $x$  is written in scientific notation if it is expressed as

$$x = a \times 10^n \quad \text{where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$$

**Definition of  $n$ th root:**

$$\sqrt{a} = b \quad \text{means} \quad b^2 = a \text{ and } b \geq 0$$

If  $n$  is any positive integer, then

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a$$

If  $n$  is even, then we must have  $a \geq 0$  and  $b \geq 0$ .

**Properties of  $n$ th roots:**

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$\sqrt[n]{a^n} = a \quad \text{if } n \text{ is odd}$$

$$\sqrt[n]{a^n} = |a| \quad \text{if } n \text{ is even}$$

**Definition of rational exponents:**

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

## 2.4 A review of basic properties of polynomials

**Definition:** A polynomial of degree  $n$  in a single variable  $x$  can be written in the general form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad (2.1)$$

where  $a_n, a_{n-1}, \dots, a_1, a_0$  are the coefficients, with  $a_n \neq 0$ .

The simplest polynomial is a constant ( $n = 0$ ) and the next simplest is a linear equation ( $n=1$ ). Let us start by examining the case with  $n = 2$ . The general polynomial for the case of  $n = 2$  is

$$f(x) = ax^2 + bx + c, \quad (2.2)$$

where  $a \neq 0$ ,  $b$ , and  $c$  are the coefficients. The graph of this function is a parabola, which we'll demonstrate shortly. We start by finding the root of this equation by completing the square in order to derive the quadratic formula.

### 2.4.1 Completing the square

Solve for the roots of the general quadratic equation

$$ax^2 + bx + c = 0, \quad \text{where } a \neq 0. \quad (2.3)$$

**Solution:** Let's start by solving a special case. Suppose that  $b = 0$ , then we could use the fundamental operations from basic algebra straight away to solve for  $x$  as shown below:

$$\begin{aligned} & ax^2 + c = 0 && (2.4a) \\ \xrightarrow{-c} & ax^2 = -c \\ \xrightarrow{\div(a \neq 0)} & x^2 = -\frac{c}{a} \\ \xrightarrow{\div\sqrt{\quad}} & x = \pm \sqrt{-\frac{c}{a}} && (2.4b) \end{aligned}$$

The solutions are a pair of real numbers provided that  $c/a < 0$ .

Next, suppose  $y$  is a function of  $x$  and we were somehow able to rewrite equation (2.3) in the form:

$$a[y(x)]^2 + c = 0.$$

Could we now solve for  $x$ ? We could certainly solve for  $y$  by repeating the above steps that we used to solve  $x$  in equation (2.4a). From equation (2.4b) we would have to be able to solve the equation

$$y(x) = \pm \sqrt{-\frac{c}{a}}$$

for  $x$ . Thus the answer is yes, provided that  $y = y(x)$  is a simple function of  $x$ .

We are now going to use the method of completing the square to find such a function of  $x$ . We'll start with the original equation (2.3) and rewrite it as in such a way as to isolate the terms involving  $x$  just as we did for the special case when  $b = 0$ .

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \xrightarrow{-c} \quad ax^2 + bx &= -c \\ \xrightarrow{\div(a \neq 0)} \quad x^2 + \frac{b}{a}x &= -\frac{c}{a} \end{aligned}$$

Now if we could just make the left-hand side of the equation a perfect square we'd have our special function  $y(x)$ . Let's start by assuming the simplest relation between  $x$  and  $y$ , a linear relation.

Suppose

$$y(x) = c_1x + c_2 \tag{2.6}$$

subject to the condition that the square of this relationship satisfies the equation

$$[y(x)]^2 = x^2 + \frac{b}{a}x + c_3, \tag{2.7}$$

where  $c_1$ ,  $c_2$ , and  $c_3$  are constants that we need to determine. Substituting (2.6) equation into (2.7) gives

$$(c_1x + c_2)^2 = c_1^2x^2 + 2c_1c_2x + c_2^2 = x^2 + \frac{b}{a}x + c_3 \tag{2.8}$$

Equating like-terms we immediately see that  $c_1$  must equal 1 if the quadratic terms (the  $x^2$  terms) are going to match. Substituting  $c_1 = 1$  into (2.8) yields

$$\begin{aligned} x^2 + 2c_2x + c_2^2 &= (x + c_2)^2 = x^2 + \frac{b}{a}x + c_3 \\ \xrightarrow{-x^2} \quad 2c_2x + c_2^2 &= \frac{b}{a}x + c_3 \end{aligned} \tag{2.9}$$

Next, equating the coefficients of the  $x$ -terms yields

$$2c_2 = \frac{b}{a} \quad \xrightarrow{\div 2} \quad c_2 = \frac{b}{2a}.$$

Lastly, equating the constant coefficients yields

$$c_3 = c_2^2.$$

We have now determined all of the coefficients:

$$\begin{cases} c_1 = 1 \\ c_2 = \frac{b}{2a} \\ c_3 = \left(\frac{b}{2a}\right)^2 \end{cases} \Rightarrow x^2 + \frac{b}{a}x + c_3 = (x + c_2)^2 = \left(x + \frac{b}{2a}\right)^2 \quad (2.10)$$

It is now clear how to proceed. We repeat the above argument for clarity

$$\begin{aligned} & ax^2 + bx + c = 0 \\ \xrightarrow{-c} & ax^2 + bx = -c \\ \xrightarrow{\div(a \neq 0)} & x^2 + \frac{b}{a}x = -\frac{c}{a} \\ \xrightarrow{+\left(\frac{b}{2a}\right)^2} & x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \end{aligned}$$

Next, complete the square on the left-hand side of equation and expand the terms on the right-hand side of the equation

$$\xrightarrow{\text{rewrite terms}} \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Use a common denominator to combine the terms on the right-hand side (RHS) of the equation

$$\begin{aligned} \xrightarrow{\text{combine terms on RHS}} & \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \\ \xrightarrow{\sqrt{\quad}} & \left(x + \frac{b}{2a}\right) = \pm \sqrt{\frac{b^2 - 4ac}{(2a)^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ \xrightarrow{-\frac{b}{2a}} & x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Ladies and gentlemen, I give you the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (2.11)$$

## 2.4.2 Completing the square to graph a parabola

Motivation: Start with  $y = ax^2$ , then goto  $y = ax^2 + c$ , and finally use completing the square to reduce, under a change of variables, the equation  $y = ax^2 + bx + c$  to  $Y = aX^2$ .

We now use the method of completing the square to express a general parabola of the form  $y = ax^2 + bx + c$  into a parabola of the form  $Y = aX^2 + C$  under the change of variables:

$$\begin{cases} X = x + \frac{b}{2a} = x - \left(-\frac{b}{2a}\right) \\ Y = \left(\frac{b^2 - c}{4a}\right) \end{cases} \quad (2.12)$$

These equations give the location of the vertex of the parabola:

$$(x_{\text{vertex}}, y_{\text{vertex}}) = \left(-\frac{b}{2a}, \frac{b^2 - c}{4a}\right). \quad (2.13)$$

We now derive this formula for the vertex of a general parabola starting from the most general equation of a parabola.

$$y = ax^2 + bx + c \quad (2.14)$$

Rearrange terms to prepare for completing the square on the right-hand side (RHS) of the equation.

$$\begin{aligned} &\xrightarrow{\text{rearrange terms}} y = a \left( x^2 + \frac{b}{a}x + \underline{\hspace{1cm}} \right) + c - a(\underline{\hspace{1cm}}) \\ &\xrightarrow{\text{add and subtract the same term}} y = a \left( x^2 + \frac{b}{a}x + \underbrace{\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2}_{\hspace{1cm}} \right) + c - a \underbrace{\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2}_{\hspace{1cm}} \end{aligned}$$

**Recall:** An equation is like a balance. If we add a term to one side of an equation and fail to add it to the other side, then the equation will be out of balance. Equivalently, we could also add and subtract the same amount on one side of the equation and still keep it in balance. This would amount to putting a 5 kg weight on one side of a balance scale and then removing it again. While this might seem ridiculous in real-world applications, in mathematics it is a powerful trick! Notice that in order to keep the equation balanced we must add and subtract the same amount on the RHS of the equation. This includes multiplying by  $a$ . A common mistake is to forget to multiply by the multiplier  $a$  since it lies outside the parentheses where the term that completes the square is added.

$$\xrightarrow{\text{complete the square}} y = a \left( x + \frac{b}{2a} \right)^2 + c - a \left( \frac{b^2}{4a^2} \right)$$



Combine the last two terms using a common denominator

$$\begin{aligned} &\xrightarrow{\text{simplify}} y = a \left( x + \frac{b}{2a} \right)^2 + \left( \frac{c - b^2}{4a} \right) \\ &\xrightarrow{\text{rearrange terms}} y = a \left( x + \frac{b}{2a} \right)^2 - \left( \frac{b^2 - c}{4a} \right) \end{aligned} \quad (2.15)$$

Notice that if we set  $y = 0$  in equation (2.15), then upon solving for  $x$  we get the quadratic formula.

### 2.4.3 Geometric proof for the pythagorean theorem

**Hint:** Make a square with side lengths  $a + b$ . Draw in the length  $c$  four times and look at the divided areas in two different ways.

### 2.4.4 Factoring Trinomials

We consider factoring trinomials of the form  $ax^2 + bxy + cy^2$  with integer coefficients and  $a \neq 0$ . Notice as a special case that if we set  $y = 1$  we have the standard trinomial  $ax^2 + bx + c$  in  $x$ . The goal is to write  $ax^2 + bxy + cy^2 = (m_1x + n_1y)(m_2x + n_2y)$ , where  $m_1, m_2, n_1, n_2$  are all integers. This is known as *factoring the polynomial*. Polynomials that cannot be factored in this way are said to be *prime*. There is an easy test to check if a trinomial of the form  $ax^2 + bx + c$  can be factored or if it is prime.

**The Prime Test:** If  $b^2 - 4ac = n^2$ , for some integer  $n$ , then the trinomial can be factored; otherwise it is prime.

**Comment1:** By factoring the polynomial, we mean writing it as a product of linear functions of the form  $mx + ny$  where  $m$  and  $n$  are integers.

**Comment2:**  $ax^2 + bx + c$  can always be written as a product of linear functions.

$$ax^2 + bx + c = a \left( x + \frac{b + \sqrt{b^2 - 4ac}}{2a} \right) \left( x + \frac{b - \sqrt{b^2 - 4ac}}{2a} \right),$$

where the coefficients need not be integers.

**Comment3:** The prime test does not tell us how to factor the polynomial. It only tells us when it can be factored.

### 2.4.5 Special products and factoring formulas

#### Important special products that show up often:

If  $a, b, c, d, x$ , and  $y$  are real numbers, then:

1.  $(a - b)(a + b) = a^2 - b^2$
2.  $(a + b)^2 = a^2 + 2ab + b^2$
3.  $(a - b)^2 = a^2 - 2ab + b^2$
4.  $(a + b)(a^2 - ab + b^2) = a^3 + b^3$
5.  $(a - b)(a^2 + ab + b^2) = a^3 - b^3$
6.  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
7.  $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
8.  $(ax + by)(cx + dy) = acx^2 + (ad + bc)xy + bdy^2$

We now give a more detailed breakdown.

#### Special product formulas:

If  $A$  and  $B$  are any real numbers or algebraic expressions, then

$$\begin{aligned} (A + B)(A - B) &= A^2 - B^2 \\ (A + B)^2 &= A^2 + 2AB + B^2 \\ (A - B)^2 &= A^2 - 2AB + B^2 \\ (A + B)^3 &= A^3 + 3A^2B + 3AB^2 + B^3 \\ (A - B)^3 &= A^3 - 3A^2B + 3AB^2 - B^3 \end{aligned}$$

#### Special factoring formulas:

$$\begin{aligned} A^2 - B^2 &= (A - B)(A + B) \\ A^2 + 2AB + B^2 &= (A + B)^2 \\ A^2 - 2AB + B^2 &= (A - B)^2 \\ A^3 - B^3 &= (A - B)(A^2 + AB + B^2) \\ A^3 + B^3 &= (A + B)(A^2 - AB + B^2) \end{aligned}$$

#### Simplifying rational expressions:

$$\frac{AC}{BC} = \frac{A}{B}$$

#### Multiplying and dividing rational expressions:

$$\begin{aligned} \frac{A}{B} \cdot \frac{C}{D} &= \frac{AC}{BD} \\ \frac{A}{B} \div \frac{C}{D} &= \frac{A}{B} \cdot \frac{D}{C} \end{aligned}$$

**Adding and subtracting rational expressions:**

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$$

**Properties of equality:**

$$A = B \Leftrightarrow B = A$$

$$A = B \Leftrightarrow CA = CB \quad (C \neq 0)$$

**Linear equations:**

A linear equation in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where  $a$  and  $b$  are real numbers and  $x$  is the variable.

**Solving a simple  $n$ th-degree equation:**

The equation  $X^n = a$  has the solution

$$X = \sqrt[n]{a} \quad \text{if } n \text{ is odd}$$

$$X = \pm \sqrt[n]{a} \quad \text{if } n \text{ is even and } a \geq 0$$

If  $n$  is even and  $a < 0$ , the equation has no real solution.

**Quadratic equations:**

A quadratic equation is an equation of the form

$$ax^2 + bx + c = 0$$

where  $a$ ,  $b$ , and  $c$  are real numbers with  $a \neq 0$ .

**Zero-product property:**

$$AB = 0 \quad \text{if and only if} \quad A = 0 \text{ or } B = 0$$

**Completing the square:**

To make  $x^2 + bx$  a perfect square, add  $(\frac{b}{2})^2$ , the square of half the coefficient of  $x$ . This gives the perfect square

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

To make a perfect square of  $ax^2 + bx$ , divide by  $a$  first

$$ax^2 + bx = x^2 + \frac{b}{a}x$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$$

**The quadratic formula:**

The roots of the quadratic equation  $ax^2 + bx + c = 0$ , where  $a \neq 0$ , are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Deriving the quadratic formula (the short version)**

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**The discriminant:**

The discriminant of the general quadratic  $ax^2 + bx + c = 0$  ( $a \neq 0$ ) is

$$D = b^2 - 4ac$$

If  $D > 0$ , then the equation has two distinct real solutions.

If  $D = 0$ , then the equation has exactly one real solution.

If  $D < 0$ , then the equation has no real solution.

### 2.4.6 Practice problems with multiplication of polynomials

Determine the degree of the polynomial in  $x$ .

- |                    |                        |                        |
|--------------------|------------------------|------------------------|
| 1. $1 + x^5$       | 5. $9$                 | 8. $\frac{x^2 + 3}{x}$ |
| 2. $(1 + x)^2$     | 6. $3x^{1/2} + 5x + 1$ | 9. $x^{20} - x^5$      |
| 3. $3x^2 + 5x + 1$ | 7. $\frac{x^2 + x}{x}$ | 10. $x^2y + 5$         |
| 4. $6x + 7$        |                        |                        |

In problems 11-15, perform the indicated operations.

11.  $(-5x + 1) + (7x + 11)$
12.  $(7x^2 - x + 9) + (11x^2 + 2x - 4)$
13.  $(3z^2 + 7z + 5) - (-z^2 - 3z + 2)$
14.  $(n^3 - 5n^2 + 3n + 4) - (-2n^3 + 4n - 3)$
15.  $(3x^2 - xy + y^2 - 2x + y - 5) + (7x^2 + 2y^2 - x + 2) - (8x^2 + 4xy - 3y^2 - x)$

In problems 16-21, use the properties of exponents to simplify each expression.

- |                          |                  |                                      |
|--------------------------|------------------|--------------------------------------|
| 16. $(3x^4)(2x^3)$       | 18. $(t^4)^{12}$ | 20. $(5a^2b)^2(3ab^2)^3$             |
| 17. $r^{7n}r^{2n}r^{4n}$ | 19. $(u^n)^{5n}$ | 21. $[2(x + 3y)^2]^5[3(x + 3y)^4]^3$ |

In problems 22-27, expand each product.

- |                                  |   |
|----------------------------------|---|
| 22. $5x^3y(3x - 4xy + 4z)$       | 25. $(5c + d)(5c - d)(25c^2 + d^2)$                   |
| 23. $(3x + 2y)(-4x + 5y)$        | 26. $(x^2 + 7xy - y^2)(3x^2 - xy + y^2 - 2x - y + 2)$ |
| 24. $(x + 2y)(x^2 - 2xy + 4y^2)$ | 27. $(4p^2q - 3pq^2 + 5pq)(p^2q - 2pq^2 - 3pq)$       |

In problems 28-31, use the appropriate special product formula (e.g.  $a^2 - b^2 = (a - b)(a + b)$ ) to perform each multiplication.

- |                          |  |
|--------------------------|--|
| 28. $[(4t^2 + 1) - s]^2$ | 30. $[(2a - 3b) - 4c][(2a - 3b) + 4c]$ |
| 29. $(3r - 2s)(3r + 2s)$ | 31. $(2 + t)(4 - 2t + t^2)$            |

**Answers to "Multiplication of Polynomials" problems:**

1. 5
2. 2
3. 2
4. 1
5. 0
6. not a polynomial in  $x$
7. 1
8. not a polynomial in  $x$
9. 20
10. not a polynomial in  $x$
11.  $2x + 12$
12.  $18x^2 + x + 5$
13.  $4z^2 + 10z + 3$
14.  $3n^3 - 5n^2 - n + 7$
15.  $2x^2 - 5xy + 6y^2 - 2x + y - 3$
16.  $6x^7$
17.  $r^{13n}$
18.  $t^{48}$
19.  $u^{5n^2}$
20.  $675a^7b^8$
21.  $864(x + 3y)^{22}$
22.  $15x^4y - 20x^4y^2 + 20x^3yz$
23.  $-12x^2 + 7xy + 10y^2$
24.  $x^3 + 8y^3$
25.  $625c^4 - d^4$
26.  $3x^4 + 20x^3y - 9x^2y^2 + 8xy^3 - 15x^2y - 5xy^2 + 14xy - 2x^3 + 2x^2 - y^4 + y^3 - 2y^2$
27.  $4p^4q^2 - 11p^3q^3 - 7p^3q^2 + 6p^2q^4 - p^2q^3 - 15p^2q^2$
28.  $16t^4 + 8t^2 + 1 - 8st^2 - 2s + s^2$
29.  $9r^2 - 4s^2$
30.  $4a^2 - 12ab + 9b^2 - 16c^2$
31.  $8 + t^3$

## 2.5 Properties of complex numbers

### Definition of complex numbers:

A complex number is an expression of the form

$$a + bi$$

where  $a$  and  $b$  are real numbers and  $i^2 = -1$ . The real part of this complex number is  $a$  and the imaginary part is  $b$ . Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

### Adding, subtracting, and multiplying complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

### Dividing complex numbers:

$$\frac{a + bi}{c + di} = \left( \frac{a + bi}{c + di} \right) \left( \frac{c - di}{c - di} \right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

### Square roots of negative numbers:

If  $-r$  is negative, then the principal square root of  $-r$  is

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of  $-r$  are  $i\sqrt{r}$  and  $-i\sqrt{r}$ .

## 2.6 Working with inequalities

### Rules for inequalities:

$$A \leq B \Leftrightarrow A + C \leq B + C$$

$$A \leq B \Leftrightarrow A - C \leq B - C$$

$$\text{If } C > 0, \text{ then: } A \leq B \Leftrightarrow CA \leq CB$$

$$\text{If } C < 0, \text{ then: } A \leq B \Leftrightarrow CA \geq CB$$

$$\text{If } A > 0 \text{ and } B > 0, \text{ then: } A \leq B \Leftrightarrow \frac{1}{A} \geq \frac{1}{B}$$

$$\text{If } A \leq B \text{ and } C \leq D, \text{ then: } A + C \leq B + D$$

**The sign of a product or quotient:**

If a product or a quotient has an even number of negative factors, then its value is positive.

If a product or a quotient has an odd number of negative factors, then its value is negative.

**Absolute value equations:**

$$|x| = C \text{ is equivalent to } x = \pm C$$

**Absolute value inequalities:**

$ x  < c$	is equivalent to	$-c < x < c$
$ x  \leq c$	is equivalent to	$-c \leq x \leq c$
$ x  > c$	is equivalent to	$x < -c \text{ or } c < x$
$ x  \geq c$	is equivalent to	$x \leq -c \text{ or } c \leq x$

**2.7 Distance, midpoints, and intercepts****Distance formula:**

The distance between the points  $A = (x_1, y_1)$  and  $B = (x_2, y_2)$  in the plane is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Midpoint formula:**

The midpoint of the line segment from  $A = (x_1, y_1)$  to  $B = (x_2, y_2)$  is

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

**Definition of intercepts:**

The  $x$ -intercepts are the  $x$ -coordinates of the points where the graph of an equation intersects the  $x$ -axis. Find them by setting  $y = 0$  and solving for  $x$ .

The  $y$ -intercepts are the  $y$ -coordinates of the points where the graph of an equation intersects the  $y$ -axis. Find them by setting  $x = 0$  and solving for  $y$ .



## 2.8 Variation/Proportional-to

### Direct variation:

If the quantities  $x$  and  $y$  are related by an equation

$$y = kx$$

for some constant  $k \neq 0$ , we say that  $y$  varies directly with  $x$ , or that  $y$  is directly proportional to  $x$ . The constant  $k$  is called the constant of proportionality.

### Inverse variation:

If the quantities  $x$  and  $y$  are related by the equation

$$y = \frac{k}{x}$$

for some constant  $k \neq 0$ , we say that  $y$  is inversely proportional to  $x$ , or that  $y$  varies inversely as  $x$ . The constant  $k$  is called the constant of proportionality.

### Joint variation:

If the quantities  $x$ ,  $y$ , and  $z$  are related by the equation

$$z = kxy$$

where  $k$  is a nonzero constant, we say that  $z$  varies jointly as  $x$  and  $y$ , or that  $z$  is jointly proportional to  $x$  and  $y$ .

## 2.9 A review of lines

### Slope of a line:

The slope of a nonvertical line that passes through the points  $A = (x_1, y_1)$  and  $B = (x_2, y_2)$  is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

### Point-slope form of the equation of a line:

An equation of the line that passes through the point  $(x_1, y_1)$  and has slope  $m$  is

$$y - y_1 = m(x - x_1)$$

### Slope-intercept form of the equation of a line:

An equation of the line that has slope  $m$  and  $y$ -intercept  $b$  is

$$y = mx + b$$

**Vertical and horizontal lines:**

An equation of the vertical line through  $(a, b)$  is  $x = a$ .

An equation of the horizontal line through  $(a, b)$  is  $y = b$ .

**General equation of a line:**

The graph of every *linear equation*

$$Ax + By + C = 0$$

is a line. Conversely, every line is the graph of a linear equation.

**Parallel lines:**

Two nonvertical lines are parallel if and only if they have the same slope.

**Perpendicular lines:**

Two lines with slopes  $m_1$  and  $m_2$  are perpendicular if and only if  $m_1m_2 = -1$ , that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

## 2.10 A review of the properties of graphs and functions

Recall that a graph is nothing but a collection of points. However, a function is a graph, but a graph need not be a function. For example, a circle is an example of a graph that is not a function because it fails the vertical line test. If we cut the circle in half with a horizontal line, then each semi-circle is a function.

**Equation of a circle:**

An equation of a circle with center  $(h, k)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the *standard form* for the equation of a circle.

If the center of the circle is the origin  $(0, 0)$ , then the equation is

$$x^2 + y^2 = r^2$$

**Definition of symmetry for a graph:**

Symmetry with respect to the  $x$ -axis: The equation is unchanged when  $y$  is replaced by  $-y$ . The graph is unchanged when reflected in the  $x$ -axis.

Symmetry with respect to the  $y$ -axis: The equation is unchanged when  $x$  is replaced by  $-x$ . The graph is unchanged when reflected in the  $y$ -axis.

Symmetry with respect to the origin: The equation is unchanged when  $x$  is replaced by  $-x$  and  $y$  by  $-y$ . The graph is unchanged when rotated  $180^\circ$  about the origin.

**2.10.1 Basic properties of functions****Definition of a function:**

A function  $f$  is a rule that assigns to each element  $x$  in a set  $A$  exactly one element, called  $f(x)$ , in a set  $B$ .

**The graph of a function:**

If  $f$  is a function with domain  $A$ , then the graph of  $f$  is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

In other words, the graph is the set of all points  $(x, y)$  such that  $y = f(x)$ ; that is, the graph of  $f$  is the graph of the equation  $y = f(x)$ .

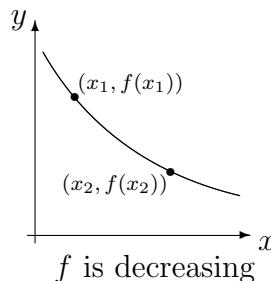
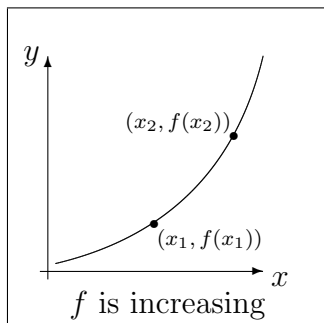
**The vertical-line test:**

A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

**Definition of increasing and decreasing functions:**

$f$  is increasing on an interval  $I$  if  $f(x_1) < f(x_2)$  whenever  $x_1 < x_2$  in  $I$ .

$f$  is decreasing on an interval  $I$  if  $f(x_1) > f(x_2)$  whenever  $x_1 < x_2$  in  $I$ .



**Average rate of change:**

The average rate of change of the function  $y = f(x)$  between  $x = a$  and  $x = b$  is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

**Shifting graphs:**

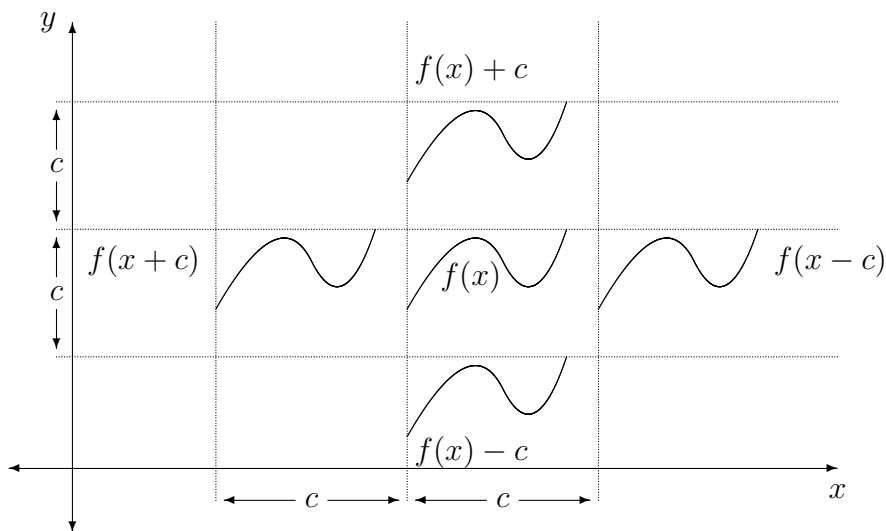
Suppose  $c > 0$ .

To graph  $y = f(x) + c$ , shift the graph of  $y = f(x)$  upward  $c$  units.

To graph  $y = f(x) - c$ , shift the graph of  $y = f(x)$  downward  $c$  units.

To graph  $y = f(x - c)$ , shift the graph of  $y = f(x)$  to the right  $c$  units.

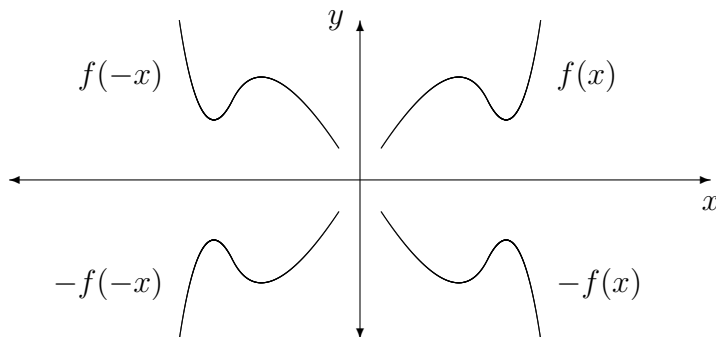
To graph  $y = f(x + c)$ , shift the graph of  $y = f(x)$  to the left  $c$  units.



**Reflecting graphs:**

To graph  $y = -f(x)$ , reflect the graph of  $y = f(x)$  in the  $x$ -axis.

To graph  $y = f(-x)$ , reflect the graph of  $y = f(x)$  in the  $y$ -axis.

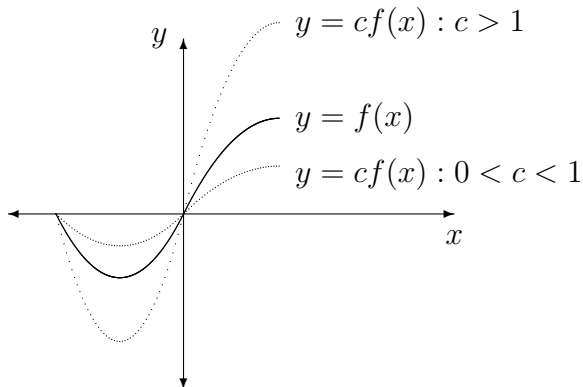


**Vertically stretching and shrinking graphs:**

To graph  $y = cf(x)$ :

If  $c > 1$ , stretch the graph of  $y = f(x)$  vertically by a factor of  $c$ .

If  $0 < c < 1$ , shrink the graph of  $y = f(x)$  vertically by a factor of  $c$ .

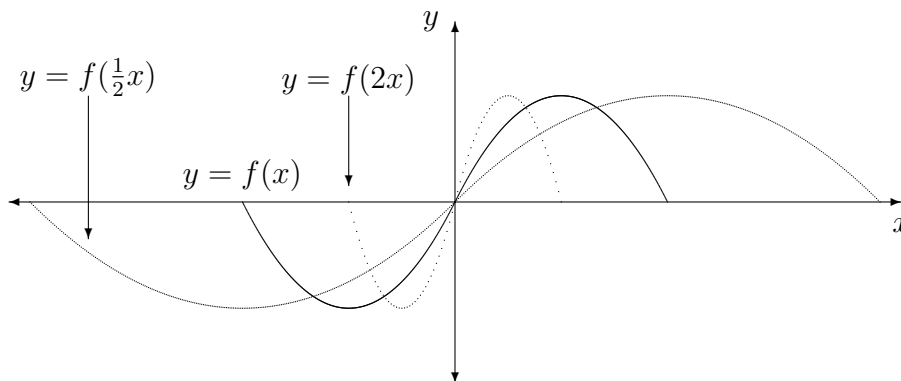


**Horizontally stretching and shrinking graphs:**

To graph  $y = f(cx)$ :

If  $c > 1$ , shrink the graph of  $y = f(x)$  horizontally by a factor of  $1/c$ .

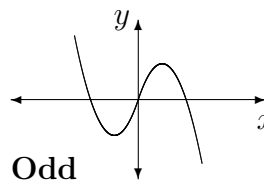
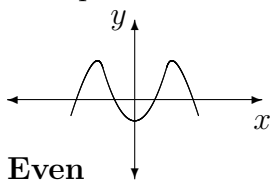
If  $0 < c < 1$ , stretch the graph of  $y = f(x)$  horizontally by a factor of  $1/c$ .



**Even and odd functions:**

A function  $f$  is even if for every  $x$  in the domain of  $f$ ,  $f(-x) = f(x)$ . The graph of an even function is symmetric with respect to the  $y$ -axis.

A function  $f$  is odd if for every  $x$  in the domain of  $f$ ,  $f(-x) = -f(x)$ . An odd function is symmetric with respect to the origin.

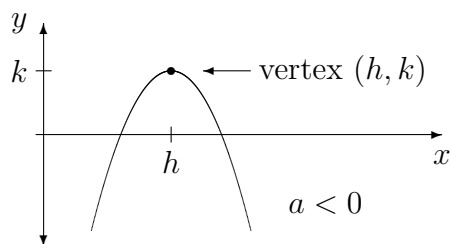
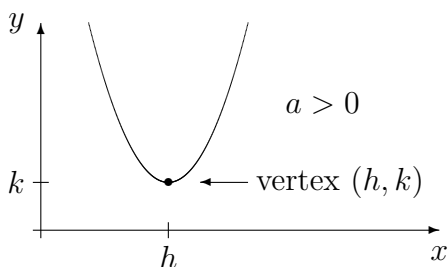


**Standard form of a quadratic equation:**

A quadratic function  $f(x) = ax^2 + bx + c$  can be expressed in the standard form

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of  $f$  is a parabola with vertex  $(h, k)$ . The parabola opens upward if  $a > 0$ , or downward if  $a < 0$ .

**Maximum or minimum value of a quadratic function:**

Let  $f(x) = a(x - h)^2 + k$  be a quadratic function in standard form.

If  $a > 0$ , then the minimum value of  $f$  occurs at  $x = h$ , and the minimum value is  $f(h) = k$ .

If  $a < 0$ , then the maximum value of  $f$  occurs at  $x = h$ , and the maximum value is  $f(h) = k$ .

If a quadratic function has the form  $f(x) = ax^2 + bx + c$ , then the maximum or minimum value occurs at

$$x = h = -\frac{b}{2a}$$

If  $a > 0$ , then  $k = f(-b/2a)$  is the minimum value.

If  $a < 0$ , then  $k = f(-b/2a)$  is the maximum value.

**Algebra of functions:**

Let  $f$  be a function with domain  $A$ , and  $g$  a function with domain  $B$ . Then we define the functions  $f + g$ ,  $f - g$ ,  $fg$ , and  $f/g$ :

Function	Domain
$(f + g)(x) = f(x) + g(x)$	$A \cap B$
$(f - g)(x) = f(x) - g(x)$	$A \cap B$
$(fg)(x) = f(x)g(x)$	$A \cap B$
$(f/g)(x) = f(x)/g(x)$	$\{x \in A \cap B \mid g(x) \neq 0\}$

**Composition of functions:**

If  $f$  and  $g$  are functions, the composition of  $f$  and  $g$  is defined as

$$(f \circ g)(x) = f(g(x))$$

**One-to-one functions:**

A function  $f$  with domain  $A$  is one-to-one if no two elements in  $A$  have the same image.

$$\text{If } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)$$

A function is one-to-one if and only if it passes the horizontal-line test: no horizontal line intersects its graph more than once.

**Inverse of a function:**

Suppose  $f$  is a one-to-one function with domain  $A$  and range  $B$ . Then its inverse function,  $f^{-1}$ , has domain  $B$  and range  $A$ , and is defined as

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

for every  $y \in B$ .

**Property of inverse functions:**

Let  $f$  be a one-to-one function with domain  $A$  and range  $B$ . The inverse function  $f^{-1}$  satisfies the following cancellation properties:

$$\begin{aligned} f^{-1}(f(x)) &= x && \text{for every } x \in A \\ f(f^{-1}(y)) &= y && \text{for every } y \in B \end{aligned}$$

Any function  $f^{-1}$  that satisfies these properties is the inverse of  $f$ .

**Polynomial functions:**

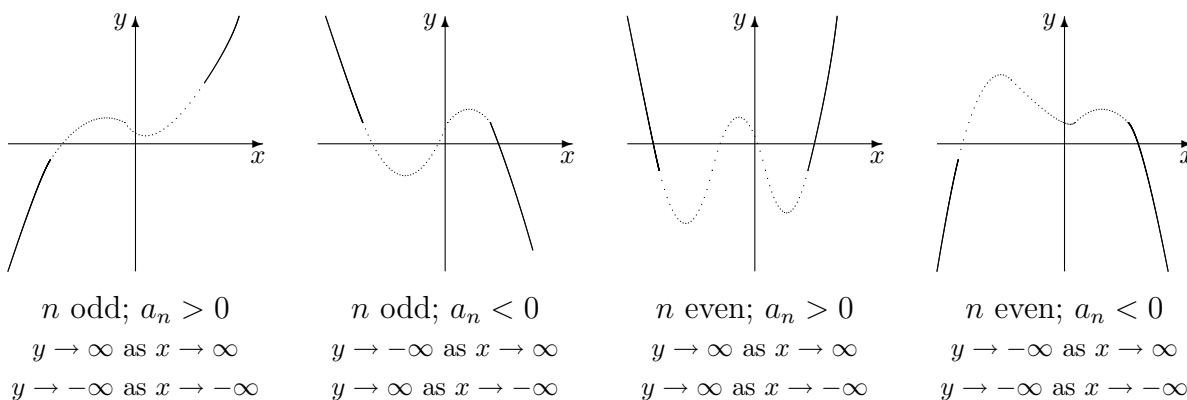
A polynomial function is a function with the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where  $n$  is a non-negative integer and  $a_0, \dots, a_n$  are real numbers, with  $a_n \neq 0$ . The numbers  $a_0, \dots, a_n$  are the coefficients of the polynomial. The number  $a_0$  is called the constant term; the number  $a_n$  is called the leading coefficient, and  $a_n x^n$  is called the leading term. The integer  $n$  is the degree of the polynomial, which can be referred to as an  $n$ th-degree polynomial.

**End behavior of a polynomial:**

The end behavior of a polynomial describes how it behaves as  $x \rightarrow \pm\infty$ . End behavior is determined by the sign of the leading coefficient  $a_n$ , and by whether the degree  $n$  of the polynomial is even or odd.



**Real zeros of polynomials:**

Suppose  $P(x)$  is a polynomial, and  $a$  is a real number. Then the following statements are equivalent:

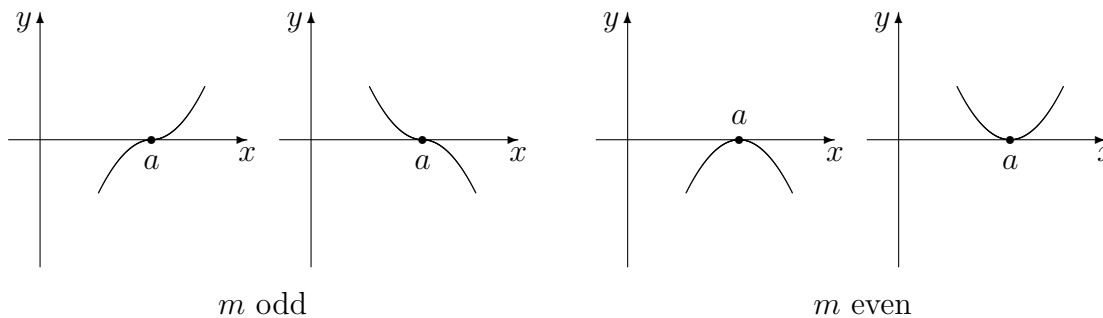
1.  $P(a) = 0$
2.  $a$  is a zero of  $P(x)$
3.  $x = a$  is a solution of  $P(x) = 0$
4.  $x - a$  is a factor of  $P(x)$
5.  $x = a$  is an  $x$ -intercept of the graph of  $P(x)$

**Intermediate-value theorem for polynomials:**

If  $P(x)$  is a polynomial, if  $a$  and  $b$  are real numbers with  $a < b$ , and if  $P(a)$  and  $P(b)$  have opposite signs, then there exists some real number  $c$  such that  $a < c < b$  and  $P(c) = 0$ .

**Behavior of the graph of a polynomial near a zero:**

Suppose  $P(x)$  is a polynomial, and  $a$  is a zero of multiplicity  $m > 1$ . Then the shape of the graph of  $P(x)$  near  $a$  is determined by whether  $m$  is even or odd. If  $m$  is odd, then the graph crosses the  $x$ -axis. If  $m$  is even, the graph is tangent to the  $x$ -axis at  $x = a$ .



**Local extrema of polynomials:**

Suppose  $P(x) = a_nx^n + \dots + a_1x + a_0$  is a polynomial of degree  $n$ . Then the graph of  $P(x)$  has no more than  $n - 1$  local extrema.



**Division of polynomials:**

If  $P(x)$  and  $D(x)$  are polynomials and  $D(x) \neq 0$ , then there are unique polynomials  $Q(x)$  and  $R(x)$ , with either  $R(x) = 0$  or the degree of  $R(x)$  less than the degree of  $D(x)$ , such that

$$P(x) = Q(x) \cdot D(x) + R(x)$$

$P(x)$  is called the dividend;  $D(x)$  is the divisor;  $Q(x)$  is the quotient; and  $R(x)$  is the remainder.

**Remainder theorem:**

If  $P(x)$  is a polynomial, and if it is divided by  $x - a$ , then the remainder is the value of  $P(a)$ .

**Factor theorem:**

If  $P(x)$  is a polynomial, then  $P(a) = 0$  if and only if  $x - a$  is a factor of  $P(x)$ .

**Rational zeros theorem:**

Suppose  $P(x) = a_n x^n + \cdots + a_1 x + a_0$  is a polynomial, and all of the coefficients  $a_0, \dots, a_n$  are integers. Then every rational zero of  $P(x)$  can be expressed as  $p/q$ , where  $p$  is a factor of the constant term  $a_0$ , and  $q$  is a factor of the leading coefficient  $a_n$ .

**Descartes' rule of signs:**

Suppose  $P(x)$  is a polynomial with real coefficients. Then the number of positive real zeros of  $P(x)$  is either equal to the number of variations in sign of  $P(x)$ , or is less than the number of variations by some even number. The number of negative real zeros is also either equal to the number of variations in sign of  $P(x)$ , or is less than the number of variations by some even number.

**Upper and lower bounds theorem:**

Suppose  $P(x)$  is a polynomial with real coefficients; and that  $a < 0$  and  $b > 0$ . Then the following are true:

1. If we divide  $P(x)$  by  $x - a$  using synthetic division, and if the row containing the quotient and remainder has alternating nonpositive and nonnegative entries, then  $a$  is a lower bound for all real zeros of  $P(x)$ .
2. If we divide  $P(x)$  by  $x - b$  using synthetic division, and if there are no negative numbers in the row containing the quotient and the remainder, then  $b$  is an upper bound for all real zeros of  $P(x)$ .

**Fundamental theorem of algebra:**

Let  $P(x) = a_n x^n + \cdots + a_1 x + a_0$  be a polynomial with complex coefficients. Then  $P(x)$  has at least one complex zero.

**Complete factorization theorem:**

Suppose  $P(x)$  is a polynomial of degree  $n > 0$ . Then there are complex numbers  $a, c_1, c_2, \dots, c_n$  such that

$$P(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$$

**Zeros theorem:**

Let  $P(x)$  be a polynomial of degree  $n$ . Then  $P(x)$  has exactly  $n$  zeros, assuming that a zero of multiplicity  $m$  is counted  $m$  times.

**Conjugate zeros theorem:**

Let  $P(x)$  be a polynomial with real coefficients. If the complex number  $z$  is a zero of  $P(x)$ , then so is its complex conjugate  $\bar{z}$ .

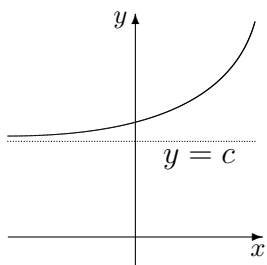
**Linear and quadratic factors theorem:**

Let  $P(x)$  be a polynomial with real coefficients. Then  $P(x)$  can be factored into a product of linear and irreducible quadratic factors with real coefficients.

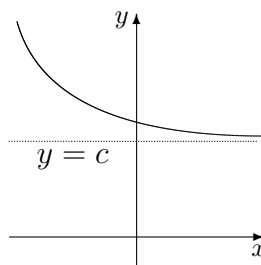
**Horizontal and vertical asymptotes:**

A line  $y = c$  is a horizontal asymptote of a function  $y = f(x)$  if:

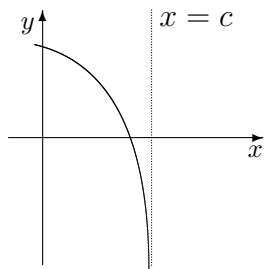
$$f(x) \rightarrow c \text{ as } x \rightarrow -\infty$$



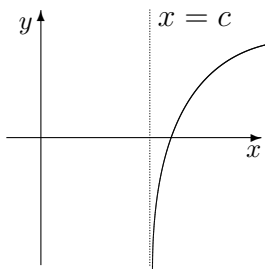
$$f(x) \rightarrow c \text{ as } x \rightarrow \infty$$



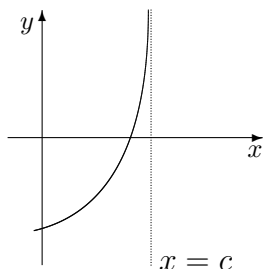
A line  $x = c$  is a vertical asymptote of a function  $y = f(x)$  if one of the following occurs:



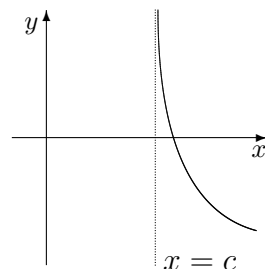
$$y \rightarrow -\infty \text{ as } x \rightarrow c^-$$



$$y \rightarrow -\infty \text{ as } x \rightarrow c^+$$



$$y \rightarrow \infty \text{ as } x \rightarrow c^-$$



$$y \rightarrow \infty \text{ as } x \rightarrow c^+$$

**Asymptotes of a rational function:**

Let  $f(x)$  be a rational function:

$$f(x) = \frac{a_mx^m + \cdots + a_1x + a_0}{b_nx^n + \cdots + b_1x + b_0}$$

If  $m < n$ , then  $f(x)$  has a horizontal asymptote at  $y = 0$ .

If  $m = n$ , then  $f(x)$  has a horizontal asymptote at  $y = a_n/b_m$ .

If  $m > n$ , then  $f(x)$  has no horizontal asymptotes.

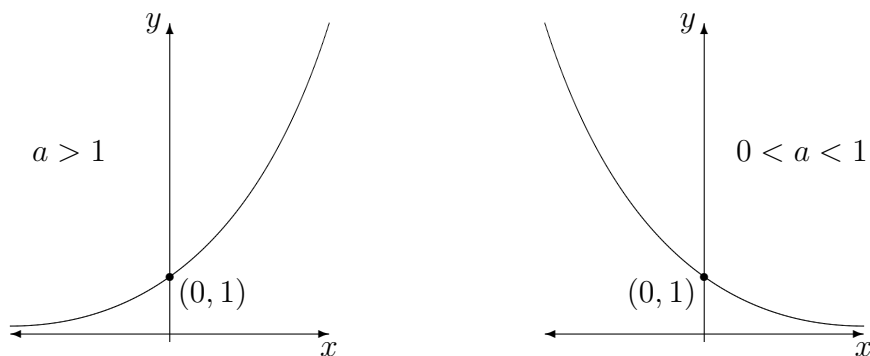
$f(x)$  has a vertical asymptote at  $x = c$  if and only if  $c$  is a zero of the denominator.

**2.11 Exponentials and Logarithms****Exponential functions:**

An exponential function with base  $a$  is defined as

$$f(x) = a^x$$

The domain of this function is  $(-\infty, \infty)$ . We require  $a > 0$  and  $a \neq 1$ . Depending on  $a$ , the graph has one of two shapes:

**Natural exponential function:**

The natural exponential function is the exponential function with base  $e$ :

$$f(x) = e^x$$

**Compound interest:**

Compound interest is calculated using

$$A(t) = P \left( 1 + \frac{r}{n} \right)^{nt},$$

where  $A(t)$  is the value of the account after  $t$  years;  $P$  is the principal;  $r$  is the annual interest rate;  $n$  is the number of times the interest is compounded in a year; and  $t$  is the time, expressed in years.

If interest is compounded continuously, then the formula is

$$A(t) = Pe^{rt}$$

**Logarithms:**

Suppose  $a > 0$  and  $a \neq 1$ . If  $x$  is a positive number, then the logarithm of  $x$  to the base  $a$  is the power to which  $a$  must be raised to equal  $x$ :

$$\log_a x = y \iff a^y = x$$

For all valid values of  $a$ , the following properties hold:

$$\log_a 1 = 0$$

$$\log_a a = 1$$

$$\log_a a^x = x$$

$$a^{\log_a x} = x$$

The logarithm to base 10 is called the common logarithm. If the base  $a$  is not written, we generally mean the common logarithm:

$$\log x = \log_{10} x$$

**Natural logarithms:**

The natural logarithm, written “ln”, is the logarithm to base  $e$ . The definition and properties are the same as they are for the logarithm to base  $a$ :

$$\ln x = \log_e x$$

$$\ln x = y \iff e^y = x$$

$$\ln 1 = 0$$

$$\ln e = 1$$

$$\ln e^x = x$$

$$e^{\ln x} = x$$

**Laws of logarithms:**

Suppose  $a \neq 0$  is a positive number, and  $A$ ,  $B$ , and  $C$  are real numbers with  $A > 0$  and  $B > 0$ . Then the following laws hold true for logarithms:

$$\log_a(AB) = \log_a A + \log_a B$$

$$\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$$

$$\log_a(A^C) = C \log_a A$$

**Change of base:**

To calculate logarithms to base  $b$  given logarithms to base  $a$ , use

$$\log_b x = \frac{\log_a x}{\log_a b}$$

## 2.12 Rules for systems of equations

**Solutions of a linear system in two variables:**

Given a system of linear equations in two variables, exactly one of the following is true:

1. The system has exactly one solution
2. The system has no solution
3. The system has infinitely many solutions

**Operations that produce an equivalent system of equations:**

1. Adding a nonzero multiple of one equation to another
2. Multiplying an equation by a nonzero constant
3. Interchanging the positions of two equations

**Number of solutions of a linear system of equations**

Given a system of linear equations, exactly one of the following is true:

1. The system has exactly one solution
2. The system has no solution
3. The system has infinitely many solutions