Question 1

In class we discussed in some detail the nonlinear Duffing oscillator:

\[ \ddot{u} + u + \epsilon u^3 = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0. \]

Carry out the Poincar-Lindstedt expansion to third order in \( \epsilon \) and show for the frequency expansion

\[ \omega = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \epsilon^3 \omega_3 + \cdots \]

that

\[ \omega_1 = \frac{3}{8}, \quad \omega_2 = -\frac{21}{256}, \quad \omega_3 = \frac{81}{2048}. \]

The time to complete a full cycle of motion is

\[ T = 2 \int_a^b \frac{du}{\sqrt{2(E - \frac{u^2}{2} - \epsilon \frac{u^4}{4})}}, \]

where \( a \) and \( b \) are the turning points of the motion. Obviously \( T \), \( a \), and \( b \) will all depend on the energy \( E \). For our particular choice of initial data, show that the period is given by

\[ T = 2 \sqrt{\frac{2}{2 + \epsilon}} K \left( -\frac{\epsilon}{2 + \epsilon} \right), \]

where \( K(m) \) is the complete elliptic integral of the first kind. Using the standard expansions available for \( K(m) \) show that a small \( \epsilon \) expansion of \( \omega = 2\pi/T \) agrees with the expansion you obtained from the Poincare-Lindstedt method.

You will need to find out about elliptic integrals for yourselves. One good source is the famous Handbook of Mathematical Functions by Abomowitz and Stegun.
Question 2

Now here is one of those silly little questions you should be asking yourselves: are the results I just obtained only good for the special choice of initial conditions I used, namely

\[ u(0) = 1, \quad \dot{u}(0) = 0, \]

or could they be generalized? In particular, what would happen if I chose some different initial conditions such as

\[ u(0) = a, \quad \dot{u}(0) = 0, \]

where \( a \neq 1 \)? Investigate this by

(i) carrying out Poincare-Lindstedt to first order,
(ii) carrying out the expansion of the frequency using the elliptic integral representation to third order. Explain, after the fact, why what you obtain is (or should have been) obvious!

Question 3

Now carry out the Poincare-Lindstedt expansion for the nonlinear oscillator

\[ \ddot{u} + u + \epsilon u^2 = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0, \]

to third order in \( \epsilon \) and find \( \omega_i \) for the frequency expansion

\[ \omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \epsilon^3 \omega_3 + \cdots \]

Comments:

- You may be in for a slight surprise when you do this.
- Although one can again write down an analytical expression for the period of motion in terms of elliptic integrals, it is messier than the Duffing oscillator and I am not asking you to do this.
- I would be quite impressed if any of you would like to try this for yourselves and obtain the small \( \epsilon \) expansion for \( \omega \) this way.

Question 4: Jordan’s voluntary question

Now carry out the Poincare-Lindstedt expansion for the nonlinearly damped oscillator

\[ \ddot{u} + u + \epsilon \dot{u}^3 = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0. \]
Note: I have not attempted to do this myself and have no idea what you will find.

Question 5

Now it’s time to take a simple example of the multiple-scales method and beat it to death. Our victim is the unsuspecting damped linear oscillator:

\[ \ddot{y} + y + \epsilon \dot{y} = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1. \]

(i) Derive (as opposed to demonstrating by direct substitution) the exact solution

\[ y(t, \epsilon) = \frac{1}{\sqrt{1 - \epsilon^2 / 4}} e^{-\epsilon t} \sin \left( t \sqrt{1 - \epsilon^2 / 4} \right) \]

and give a few plots of the solution for different values of \( \epsilon \).

(ii) Now determine the leading order multiple-scales expansion

\[ y = y_0(t_1, t_2), \quad t_1 = t, \quad t_2 = \epsilon t. \]

This is about as simple as it gets (easily done by hand). Show that

\[ y_0(t_1, t_2) = e^{-t_2/2} \sin(t_1). \]

(iii) Now carry out a multiple-scales expansion to next order in the form:

\[ y = y_0(t_1, t_2) + \epsilon y_1(t_1, t_2). \]

Warning: This is more difficult than the leading-order calculation because of issues that arise with the initial conditions. Show that

\[ y_1(t_1, t_2) = -\frac{1}{8} e^{-t_2/2} t_2 \cos(t_1). \]

(iv) Now develop a multiple-time-scales solution involving a third time scale in the form:

\[ y = y_0(t_1, t_2, t_3), \quad t_1 = t, \quad t_2 = \epsilon t, \quad t_3 = \epsilon^2 t. \]

This task is not so simple, and at a certain point in your calculation you will need to make a seemingly arbitrary assumption in order to close the calculation. At the end of the day (either metaphorically or literally) you should find (i.e., you need to show that)

\[ y_0(t_1, t_2, t_3) = e^{-t_2/2} \sin(t_1 + t_3/8). \]

Suggest, after the fact, why this is probably the right answer.
So we now have 3 possible approximations to compare with the exact solution:

\[ y^{(1)} = y_0(t_1, t_2), \quad y^{(2)} = y_0(t_1, t_2) + \epsilon y_1(t_1, t_2), \quad y^{(3)} = y_0(t_1, t_2, t_3) \]

Give some numerical studies of these three solutions. In particular, you should see that \( y^{(2)} \) and \( y^{(3)} \) are barely distinguishable. As a way of seeing how they differ, expand the difference \( y_{\text{exact}} - y^{(2)} \) and \( y_{\text{exact}} - y^{(3)} \) in a series in \( \epsilon \) and identify where the solutions start to differ.

**Question 6**

Consider the nonlinearly damped oscillator

\[ \ddot{u} + 9u - \epsilon \left( (1 - u^4)\dot{u} + u^3 \right) = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0. \]

(i) Based on physical considerations and numerical studies discuss the behavior of this system.

(ii) Find the first-order multiple-scales solution to this problem.

(iii) Now make a small change in the nonlinearity, namely

\[ \ddot{u} + 9u - \epsilon \left( (1 - u^4)\dot{u} + u^4 \right) = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0, \]

and find the first-order multiple-scales solution to this problem.