Example 0: (Bender and Orszag, pg 419 chaps 9.1 and 9.2)

\[
\begin{cases}
\varepsilon y'' + (1+\varepsilon)y' + y = 0, \\
y(0) = 0, \; y(1) = 1, \quad \text{and} \; \varepsilon > 0.
\end{cases}
\]

The exact solution is
\[
y\text{exact}(x) = \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}}.
\]

The uniform solution is
\[
y\text{uniform}(x) = \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - 0}.
\]

So the uniform solution is missing the exponentially small term \(e^{-\frac{x}{\varepsilon}}\). The reason for this is all of our expansions were of the form of a power series in \(\varepsilon\) (i.e. \(y = \sum \chi_n \varepsilon^n\)). Terms such as \(\varepsilon^n\) cannot capture exponentially small terms.

Exponentially small terms are often known as "beyond-all-orders terms".

This means that the are terms that are smaller than any term of the form \(\varepsilon^n\) as \(\varepsilon \to 0\). In particular, for our case
\[
\varepsilon^n \gg e^{-\frac{x}{\varepsilon}} \quad \text{as} \; \varepsilon \to 0^+ \quad (\forall n \in \mathbb{R}).
\]

Beyond-all-orders terms can take many forms, but they generally have some form of an exponential component.

Notice that the relative error between \(y\) and \(y\text{uniform}\) is exponentially small:

\[
\text{relative error} = \left| \frac{y\text{exact} - y\text{uniform}}{y\text{exact}} \right| = \left| 1 - \frac{y\text{uniform}}{y\text{exact}} \right| = e^{-1 - \frac{1}{\varepsilon}}
\]
Example 1 (cont.)

\[
\frac{Y_{\text{unit}}}{Y_{\text{exact}}} = \frac{e^{-x} - e^{-\frac{x}{\epsilon}}}{e^{-1} - e^{-\frac{1}{\epsilon}}} = \frac{e^{-1} - e^{-\frac{1}{\epsilon}}}{e^{-1} - e^{-\frac{1}{\epsilon}}} = 1
\]

\[\Rightarrow \text{rel error} = \left| 1 - \frac{Y_{\text{unit}}}{Y_{\text{exact}}} \right| = e^{1 - \frac{1}{\epsilon}}\]
Example 0: Consider the following BVP:

\[ \epsilon y'' - x^2 y' - y = 0, \]

\[ y(0) = y(1) = 1. \]

Solution:

Recall: Start by referencing example 3 from section 9.1:

\[ \epsilon y'' + a(x) y' + b(x) y = 0 \]

\[ y(0) = A, \quad y(1) = B \quad \text{on the interval } [0, 1]. \]

It was assumed that \( a(x) \neq 0 \) on \([0, 1]\). Under this condition, it was shown that a boundary layer (B.L.) could only occur at one of the end points, and moreover, that the B.L. would have a thickness of order \( \epsilon \) (i.e., \( \delta(\epsilon) = \epsilon \)).

If \( a(x) > 0 \), then the B.L. forms at \( x = 0 \) (the left end pt.).
If \( a(x) < 0 \), then the B.L. forms at \( x = 1 \) (the right end pt.).

For our problem we have a different situation:

\[ a(x) = -x^2 < 0 \quad \text{on } [0, 1], \]

but which equals zero at 0.

The 1st thing to notice is that there was nothing special about the end points \( x = 0 \) and \( x = 1 \) in example 3. He could have chosen any end points, say \([a_1, b_1]\), and have done a similar analysis. In particular, for \( 0 < a_1 << 1 \) and \( b_1 = 1 \) the analysis done in example 3 should hold. Thus, on \([a_1, 1]\) we have \(-x^2 < 0\), so we expect a B.L. of thickness \( \delta = \epsilon \) at \( x = 1 \).

At the left end point \( x = 0 \) we have no analysis to fall back on. We must therefore forge our own path for the analysis at the point \( x = 0 \). Since \(-x^2 < 0 \) for \( x \neq 0 \), this is the only other possible location for a B.L.