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1 First-order ODEs

1.0 Chapter overview

In this chapter we focus on first-order ODEs (i.e., differential equations with only first-order derivatives in it). We write first-order differential equations in two common forms:

\[ G(x, y, y') = 0 \]  
\[ y' = f(x, y) \]  
\[ M(x, y) \, dx + N(x, y) \, dy = 0 \]

(Most General Form)  
(Standard Normal Form)  
(Differential Form)

We begin this section with the theory of first-order initial-value problems. This will tell you when a first-order ODE has a unique solution. This is a very important step, since there is no point in searching for a solution if it doesn’t exist!

We will then move on and learn a standard collection of techniques for solving ODEs, which will start with the very simple separation of variables and move on to more advanced techniques that invoke insight.

When solving first-order ODEs, there are four types of equations that every student of differential equations should have mastered. We now list them and the techniques needed to solve these equations.

- Separable equations \( \Rightarrow \) The method of separation of variables
- Linear equations \( \Rightarrow \) Determining an integrating factor
- Exact equations \( \Rightarrow \) Forming exact differentials (primitives)
- Homogeneous equations \( \Rightarrow \) Homogeneous functions and substitution methods

Although there are many other types of equations that can also be solved analytically, a clever substitution will usually reduce them to one of the first three type of equations. At this point you will be somewhat in the dark as to how and why the methods are actually giving a solution to the equation, and for that matter what a solution to a differential equation even is. This is normal. To remedy this, we will return to these topics after you have had some practice manipulating ODEs and give a more formal treatment of the derivation of the methods and some useful theorems involving differential equations.

The next topics that we will examine are:

- Integrating Factors: the making of exact differentials
- Reduction methods: Bernoulli, Ricatti, Clairaut’s equations, and reduction of order: converting a second order ODE to a first order ODE
• Substitution methods

These are more advanced topics that require reasonable computational skills on the part of the student. The idea will always be to reduce an ODE that you can’t solve, to one that you can solve by one of the four basic methods above.

Upon completing these topics we will apply our new found techniques to a series of application problems. Having completed applications problems you will now have more experience in solving ODEs. You will have some intuition as to what a solution to an ODE actually is. It is at this point in your journey through ODEs that it makes sense to give the foundation of first order ODEs from a more formal perspective. It will be at this point when it will be most fruitful to study why the four main methods that you have been using work. We will then use our new-found understanding of the methods as well as our knowledge of the powerful existence and uniqueness theorem for first-order ODEs to move on to higher-order ODEs, which are in general much more difficult problems. But before doing this, we’ll take a slight digression to learn the core ideas behind linear algebra, which will aide us in our study of higher-order equations.

1.1 First-order initial-value problems

This chapter is concerned with the solution of the first-order equation with initial condition:
\[
\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x_0) = y_0
\]  

(1.1)

We’ll begin with a general look at some approaches to solving Equation (1.1). First, though, we should ask ourselves: is there necessarily a solution at all?

1.1.1 Existence and uniqueness of the solution to the IVP

An equation with initial condition(s) like (1.1) is called an *initial-value problem (IVP)*. Before we try to solve an IVP, there are two questions we need to answer:

1. Is there a solution?
2. Is it unique?

The following theorem gives a sufficient condition for the existence of a unique solution.
Theorem 1 (Existence and uniqueness theorem for first-order ODEs). Let \( R \) be a rectangular region in the \( xy \)-plane defined by \( a \leq x \leq b, \ c \leq y \leq d \) that contains the point \( (x_0, y_0) \) in its interior. If \( f(x, y) \) and \( \partial_y f \) are continuous on \( R \), then there is an interval \( I \) centered at \( x_0 \) and a unique function \( y(x) \) defined on \( I \) satisfying Equation (1.1). (See the figure below.)

Some comments:
1. If there is a solution to the IVP, then it must include the point \( (x_0, y_0) \).
2. The criteria for this theorem are easy to check.
3. It’s one thing to know that a solution exists; it’s another thing to find it. You will soon come to a deep understanding of this fact!
4. Knowing that there’s a unique solution is useful when using numerical methods to solve a problem. If the ODE has no solution, then there is no point in trying to find one. In the best-case scenario, you’d waste your time and realize that your numerical scheme wasn’t converging on a solution. In the worst case, you would find a false solution! Even if you know a solution exists and your numerical scheme converges to a solution, you want to be sure it’s the solution you intended to find. This is where the uniqueness aspect of the theorem is useful.

Example 1. Determine a region of the \( xy \)-plane for which the ODE \( \frac{dy}{dx} = y^{2/3} \) would have a unique solution through a point \( (x_0, y_0) \) in the region.

**Solution**: Here \( f(x, y) = y^{2/3} \) and \( f_y = \frac{2}{3}y^{-1/3} \). First, notice that \( f \) is independent of \( x \). (Functions like that are called cylinder functions, because they’re missing one or more variables.) \( f \) is continuous everywhere in the \( xy \)-plane. \( f_y \) is continuous everywhere in the \( xy \)-plane except at \( y = 0 \) (the \( x \)-axis).

Thus by Theorem 1, there is a solution passing through any point \( (x_0, y_0) \) contained in a region whose intersection with the \( x \)-axis is empty. Put another way, given any \( (x_0, y_0) \) for which \( y_0 \neq 0 \), there is a unique solution that includes \( (x_0, y_0) \) and is valid for some region of the plane. The closer \( y_0 \) is to zero, the smaller the region of validity.

Example 2. Determine a region of the \( xy \)-plane for which the ODE

\[
\frac{dy}{dx} = \sqrt{xy} = f(x, y)
\]

would have a unique solution through a point \( (x_0, y_0) \) in the region.
**Solution:** Here $f(x, y)$ is continuous on $xy \geq 0$. Its partial derivative

$$
\partial_y f = \partial_y (xy)^{1/2} = \frac{\sqrt{x} y^{-1/2}}{2} = \frac{1}{2} \frac{\sqrt{x}}{y}
$$

is continuous on $xy \geq 0$ provided $y \neq 0$. By Theorem 1, if $(x_0, y_0)$ is any point in the first quadrant ($x_0 \geq 0$, $y_0 > 0$) or the third quadrant ($x_0 \leq 0$, $y_0 < 0$), then there is a unique solution to the ODE that includes the point $(x_0, y_0)$.

**Example 3.** Determine a region of the $xy$-plane for which the ODE

$$(y - x)y' = y + x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y + x}{y - x} = f(x, y)$$

would have a unique solution through a point $(x_0, y_0)$ in the region.

**Solution:** Here $f(x, y)$ is continuous if $x \neq y$. The partial derivative

$$
f_y(x, y) = \partial_y [(y + x)(y - x)^{-1}]
$$

$$
= (y - x)^{-1} - (y - x)^{-2}(y + x) = \frac{(y - x) - (y + x)}{(y - x)^2} = \frac{-2x}{(y - x)^2}
$$

is also continuous wherever $x \neq y$. Thus the ODE will have a unique solution in any region contained in $y > x$ or in $y < x$.

**Example 4.** Determine a region of the $xy$-plane for which the ODE

$$
\frac{dy}{dx} = (x - 1)e^{y/(x-1)} = f(x, y)
$$

would have a unique solution through a point $(x_0, y_0)$ in the region.

**Solution:** Note that

$$
\lim_{x \to 1^-; \; y > 0} f(x, y) = 0 \quad \text{but} \quad \lim_{x \to 1^+; \; y > 0} f(x, y) = \infty
$$

Hence $f(x, y)$ is not continuous at $x = 1$. The partial derivative is calculated using the chain rule:

$$
f_y = \partial_y [(x - 1)e^{y/(x-1)}] = (x - 1)e^{y/(x-1)} \partial_y \left( \frac{y}{x - 1} \right) = e^{y/(x-1)}
$$

It is also not continuous at $x = 1$. Thus the ODE will have a unique solution in any region where $x \neq 1$, i.e. in the region $x > 1$ or the region $x < 1$.

**Example 5.** Consider the initial-value problem

$$
y' = |y - 1| = f(x, y) \quad \text{with} \quad y(0) = 1.
$$

(i) Verify that the hypotheses of the theorem do not hold for this problem.

(ii) By inspection only, find a solution to the IVP. Thus, even though the theorem does
not guarantee a solution to the IVP, a solution exists none the less.

**Solution:** Clearly, $y = 1$ is a solution to the IVP Since $f(x, y) = |y - 1|$ is not continuous at $y = 1$, and $\partial_y f(y)$ does not even exist at $y = 1$, let alone be continuous, we cannot apply the theorem to this case. However, although the ODE fails to satisfy the hypothesis of Theorem 1, it still has a unique solution. This is because the hypothesis of the theorem is a sufficient condition, not a necessary one, for the existence and uniqueness of a solution to the ODE.
Example 6. Consider the first-order autonomous initial-value problem

\[ \frac{dy}{dx} = 1 + y^2 \quad y(x_0) = y_0. \]

(i) For what region of the \( xy \)-plane does the theorem guarantee a unique solution?
(ii) Use the method of separation of variables to solve the equation for a general initial condition.
(iii) For the case \( y(0) = 0 \) find the largest interval containing the origin for which a unique solution exists.

Solution:

(i) Since \( f(x,y) = f(y) = 1 + y^2 \) and \( \partial_y f = f'(y) = 2y \) are continuous everywhere in the \( xy \)-plane, we know that the ODE has a unique solution everywhere.

(ii) Rearranging and integrating gives us

\[
\begin{align*}
\frac{dy}{dx} &= 1 + y^2 \\
\frac{dy}{1 + y^2} &= dx \\
\tan^{-1}(y) &= x + c \\
y &= \tan(x + c)
\end{align*}
\]

Applying the initial condition yields:

\[ y_0 = \tan(x_0 + c) \quad \arctan(\cdot) \rightarrow \quad c = \tan^{-1}(y_0) - x_0. \]

The solution is then given by

\[ y = \tan[(x - x_0) + \tan^{-1}(y_0)]. \]

Some comments are in order:

1. The general solution agrees with the solution to the solution in problem 7.
2. The general solution is \( \pi \)-periodic.
3. The general solution is not continuous over the whole real line.
4. The general solution has a singularity at every \( x = x_0 - \tan^{-1}(y_0) + \frac{\pi}{2} + n\pi \).

(iii) Applying the initial condition \( y(0) = 0 \) we find that the unique solution is \( y = \tan x \) on the interval \((-\pi/2, \pi/2)\). As a check, notice that \( y = \tan x, y' = \sec^2 x = 1 + y^2 \) and \( \tan(0) = 0 \). Because \( y = \tan x \) has singularities at \( x = \pm \frac{\pi}{2} \), the theorem only guarantees that a unique solution exists locally near \((x_0, y_0)\); it says nothing about the size of the interval of validity.
**Definition 1.** An ODE is said to be autonomous (running by itself) if the independent variable does not show up explicitly in the ODE.

For the case of a first-order ODE with independent variable $x$, the $x$ would not show up in the equation. Thus, the equation would have the form:

$$y' = f(y).$$

**Theorem 2** (Existence and uniqueness theorem for a first-order autonomous ODEs). Consider the first-order autonomous ODE together with its initial condition:

$$\dot{x} = f(x) \text{ with } x(0) = x_0.$$ 

Let $(a, b)$ be an open interval on the $x$ axis with $x_0 \in (a, b)$. If $f(x)$ and $f'(x)$ are continuous on $(a, b)$, then the initial-value problem has a unique solution on some time interval $[0, T]$.

### 1.1.2 Solving a general first-order ODE

Now, let’s return to the problem of finding solutions for Equation (1.1):

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0$$

with certain conditions on the form of the function $f(x, y)$.

First, the bad news: there is no general closed-form solution for this IVP.

If we try a straightforward approach, we will fail to arrive at a solution. However, we will gain insight into why the general solution is so elusive. Let’s assume the existence of a unique solution

$$y = \phi(x) \quad (1.2)$$

Then we can write

$$dy = \phi'(x) \, dx \quad (1.3)$$

Again, Equation (1.1) is

$$\frac{dy}{dx} = f(x, y)$$

Multiply by $dx$:

$$dy = f(x, y) \, dx$$

We’ve assumed $y = \phi(x)$ is the solution curve. We will now use this fact to integrate equation (1.1) along the solution curve (1.2). We also assume that $f$ is continuous on an interval containing $x_0$, so $f$ can be integrated from $x_0$ to $x$:

$$\int_{y(x_0) = y_0}^{y = \phi(x)} dy = \int_{x_0}^{x} f(t, \phi(t)) \, dt.$$
Simplify the left side:

\[ \phi(x) - y_0 = \int_{x_0}^{x} f(t, \phi(t)) \, dt \]

Rearrange:

\[ \phi(x) = \int_{x_0}^{x} f(t, \phi(t)) \, dt + y_0 \quad (1.4) \]

This says that the value of \( \phi \) at \( x \) depends on the value of \( \phi \) over the interval \([x_0, x]\). This type of situation usually requires some sort of iteration process.

We’ve now got an implicit relation defining \( y(x) \). Equation (1.4) is known as an integral equation. In general, there is no closed-form solution to (1.4). In fact, for a general function \( f(x, y) \), we have no guarantee that a solution even exists, let alone that it is unique. We will assume that \( f \) and \( \partial_y f \) are piecewise continuous on an interval containing \( x_0 \). Then by Theorem 1, we know that a unique solution exists.

One approach to solving Equation (1.4) is to use successive approximations to \( y(x) \). The following iterative process is known as Picard’s method; see also section 2.8.

\[
\begin{align*}
y_0(x) &= y_0 \quad \text{(starting point)} \\
y_1(x) &= \int_{x_0}^{x} f(t, y_0) \, dt + y_0 \quad \text{(first approximation)} \\
y_2(x) &= \int_{x_0}^{x} f(t, y_1(t)) \, dt + y_0 \quad \text{(second approximation)} \\
&\vdots \\
y_n(x) &= \int_{x_0}^{x} f(t, y_{n-1}(t)) \, dt + y_0 \quad \text{(n-th approximation)}
\end{align*}
\]

The hope is that in the limit as \( n \to \infty \), \( y_n(x) \) “converges” to \( y(x) \). This will be true if \( f \) satisfies certain conditions.

A second approach is to use direction fields to graphically pick out the “flow of the solution”; see section 9.1. This method is less direct and precise. Moreover, when you’ve used it, you only have a set of data points; you then have to use a curve-fitting scheme to get a solution over a continuous interval.

Let’s see how we would get an approximate graphical solution to the ODE (1.1):

\[ \frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x_0) = y_0 \]

We start with the fact that the solution includes the point \((x_0, y_0)\). We have another piece of information about the solution curve: its slope \( y'(x_0) = f(x_0, y_0) \) at \( x_0 \). Next, we
choose some small value for $\Delta x$, and let $x_1 = x_0 + \Delta x$. We can approximate $y_1 = y(x_1)$ using the tangent to the curve at $x_0$:

$$y_1 = y(x_1) = y_0 + f(x_0, y_0)\Delta x$$

We can repeat this process indefinitely:

$$\begin{align*}
x_2 &= x_1 + \Delta x \\
y_2 &= y(x_2) = y_1 + f(x_1, y_1)\Delta x \\
\vdots \\
x_n &= x_{n-1} + \Delta x \\
y_n &= y(x_n) = y_{n-1} + f(x_{n-1}, y_{n-1})\Delta x
\end{align*}$$

This diagram shows the process graphically:

Example 7. We have already seen that a solution to the ODE

$$y' = y \quad \text{with} \quad y(0) = 1$$

is $y = e^x$. In this case, $f(x, y) = y$ and $(x_0, y_0) = (0, 1)$. Let’s use the method of direction fields to find an approximate value for $y$ at $x = \frac{2}{10}$, using a step of $\Delta x = \frac{1}{10}$.

Step 1: Compute $y_1 = y_0 + f(x_0, y_0)\Delta x$, where $f(x, y) = y$.

$$y_1 = 1 + 1 \cdot \frac{1}{10} = \frac{11}{10}, \quad x_1 = x_0 + \Delta x = 0 + \frac{1}{10} = \frac{1}{10}$$

Step 2: Compute $y_2 = y_1 + f(x_1, y_1)\Delta x = y_1 + y_1\Delta x$

$$y_2 = \frac{11}{10} + \frac{11}{10} \cdot \frac{1}{10} = \frac{11}{10} \cdot \frac{11}{10} = \left(\frac{11}{10}\right)^2 = \frac{121}{100}, \quad x_2 = x_1 + \Delta x = \frac{2}{10}$$

The exact solution is $y(x) = e^x$. By a Taylor series approximation,

$$y(2/10) = e^{2/10} \approx 1 + \frac{2}{10} + \frac{1}{2} \left(\frac{2}{10}\right)^2 = \frac{122}{100}$$
This method worked well in this case because we took very few steps; because $\Delta x$ was small; because we wanted an approximate solution at a point $x$ that was close to our original $x_0$; and because the solution function was well-behaved—it didn’t oscillate wildly, etc. On the other hand, suppose we’d wanted to know the solution at $x = 100$ or $x = 10,000$. Then the method would not have worked well at all! We would need a predictor-corrector method to approximate the solution over such large distances.

The distance over which the approximation is valid depends on the problem. Among other things, it depends on how sensitive the problem is to the initial conditions. Consider the same differential equation, with two different IC’s that are “close” to one another:

\[
\frac{dy}{dx} = f(x,y) \quad \text{IC: } y(x_0) = y_0 \quad \text{Solution: } y = \phi(x; y_0)
\]

\[
\frac{dy}{dx} = f(x,y) \quad \text{IC: } y(x_0) = y_0 + \epsilon \quad \text{Solution: } y = \phi(x; y_0 + \epsilon)
\]

Here we assume that $\epsilon$ is a very small value. Nevertheless, for some functions the difference between these solutions can grow very large, even over a small interval $[x_0, x]$. Thus the smallest approximation to the equation could lead to incredibly large errors.