Homogeneous Functions

**Definition**

The function $f(x, y)$ is homogeneous of degree $n$ if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y), \text{ where } \lambda \neq 0.$$ 

**Example**

Any polynomial in $x$ and $y$ with coefficients $a_k$ of the form

$$\sum_{k=0}^{N} a_k x^k y^{n-k},$$

is homogeneous of degree $n$.

So, for example,

$$f(x, y) = x^2 + 2xy + y^2,$$

is homo. of deg. 2 since

$$f(\lambda x, \lambda y) = (\lambda x)^2 + 2(\lambda x)(\lambda y) + (\lambda y)^2$$

$$= \lambda^2 (x^2 + 2xy + y^2).$$

Notice that the exponents in each term add to two ($x^1y^1 \rightarrow 1+1 = 2$).

(Each term is of the form $x^k y^{n-k}$ has exponents that sum to $n$)

The function

$$f(x, y) = 5 + xy + y^2$$

is **not** homogeneous since

$$f(\lambda x, \lambda y) = 5 + (\lambda x)(\lambda y) + (\lambda y)^2 = 5 + \lambda^2 (xy + y^2) \neq \lambda^n f(x, y).$$

Notice that the constant term 5 keeps the function from being homogeneous.

**Fact**

The sum (or difference) of homogeneous functions of degree $n$ is again a homogeneous function of degree $n$.

**Proof**

Let $f_1(x, y), \ldots, f_K(x, y)$ all be homogeneous functions of degree $n$.

Let

$$h(x, y) = \sum_{k=1}^{K} f_k(x, y).$$

Then

$$h(\lambda x, \lambda y) = \sum_{k=1}^{K} f_k(\lambda x, \lambda y) = \sum_{k=1}^{K} \lambda^n f_k(x, y) = \lambda^n h(x, y).$$

Notice that step (1) follows from the fact that each $f_k$ is homogeneous of degree $n$. 

$$f(x, y) = x^2 + 2xy + y^2,$$

is homo. of deg. 2 since

$$f(\lambda x, \lambda y) = (\lambda x)^2 + 2(\lambda x)(\lambda y) + (\lambda y)^2$$

$$= \lambda^2 (x^2 + 2xy + y^2).$$

Notice that the exponents in each term add to two ($x^1y^1 \rightarrow 1+1 = 2$).

(Each term is of the form $x^k y^{n-k}$ has exponents that sum to $n$)
**Fact 2** If we sum a homogeneous function $f$ of degree $n$ and a homogeneous function $g$ of degree $m$, where $m \neq n$, then $f + g$ is not a homogeneous function. (Without loss of generality assume $m < n$)

**Proof** Let $h(x, y) = f(x, y) + g(x, y)$. Now $h$ is homogeneous of degree $\alpha \in \mathbb{N}$ if

$$
\forall \lambda \in \mathbb{R}, h(\lambda x, \lambda y) = \lambda^{\alpha} h(x, y)
$$

Then

$$
\Rightarrow f(\lambda x, \lambda y) + g(\lambda x, \lambda y) = \lambda^{\alpha} f(x, y) + \lambda^{\alpha} g(x, y)
$$

$$
\Rightarrow \lambda^{\alpha} (f(x, y) + g(x, y)) = \lambda^{\alpha} f(x, y) + \lambda^{\alpha} g(x, y)
$$

$$
\Rightarrow h(x, y) = \lambda^{n-\alpha} f(x, y) + \lambda^{m-\alpha} g(x, y) \quad \text{(divide by } \lambda^{\alpha})
$$

$$
\Rightarrow n - \alpha = m - \alpha
$$

$$
\Rightarrow n = m
$$

We can now understand why $x^2 + 2xy + y^2$ is homogeneous

$x^2 + 2xy + y^2 = \text{sum of homo. funs of degree 2}$

and why $5 + xy + y^2$ is not homogeneous

Thus $5 + xy + y^2$

By a similar argument all of the following polynomials are not homogeneous.

$x^3 + (xy) + y^3$, $x^2 + x^5$, $y^4x + x^5 + x^2y^2$, $x + \frac{2}{x}$

To check the degree of a term like $y^4x$ remember you must add the exponents $y^{4+1} = 5$. 

Next, we look at homogeneous functions that are not polynomials.

**Example 5:** \( f(x, y) = 2y + \sqrt{x^2 + y^2} \)

\[ f(\lambda x, \lambda y) = 2\lambda y + \sqrt{(\lambda x)^2 + (\lambda y)^2} = \lambda (2y + \sqrt{x^2 + y^2}) = \lambda f(x, y) \]

Provided \( \lambda > 0 \), we see that \( f \) is homogenous with degree \( 1 \).

\( f(x, y) = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 - y^2}} \)

\[ f(\lambda x, \lambda y) = \frac{\sqrt{(\lambda x)^2 + (\lambda y)^2}}{\sqrt{(\lambda x)^2 - (\lambda y)^2}} = \frac{|\lambda| \sqrt{x^2 + y^2}}{|\lambda| \sqrt{x^2 - y^2}} = \lambda f(x, y) \]

Thus \( f \) is homogeneous of degree 0.

\( f(x, y) = e^x \)

\[ f(\lambda x, \lambda y) = e^{\lambda x} = (f(x, y))^\lambda \neq \lambda f(x, y) \]

so \( f \) is not homogeneous.

We can't pull the \( \lambda \) out of the argument.

\( f(x, y) = e^{xy} \)

Notice that the \( \lambda \) cancels in the exponent.

\[ f(\lambda x, \lambda y) = e^{\lambda xy} = e^{xy} = f(x, y) = \lambda^0 f(x, y) \]

Homogeneous of degree 0.

\( f(x, y) = \tan(xy) \) (can't pull \( \lambda \) out and \( \lambda \) doesn't cancel out of the argument)

\( f(\lambda x, \lambda y) = \tan(\lambda x \lambda y) \)

\( f(x, y) = \ln(x) \). Clearly not homogeneous. We can't get the \( \lambda \) out of the argument.

\( f(x, y) = \ln(x) - \ln(y) \)

\[ f(\lambda x, \lambda y) = \ln(\lambda x) - \ln(\lambda y) = \ln(\lambda) - \ln(x) = \ln(\lambda) = \lambda \ln(x) = \lambda f(x, y) \]

Homogeneous of degree 0.

\( f(x, y) = \sqrt{y-x} \) (homogeneous with degree \( \frac{1}{2} \))

\( f(x, y) = \frac{xy-y^2}{x+y} \) (homogeneous, \( \deg = 1 \))

\( f(x, y) = |x| + |y| \)

\[ f(\lambda x, \lambda y) = |\lambda| f(x, y) \]
Let's return to our general homogeneous polynomial of degree \( n \). But first we look at a special case \( n=2 \).

\[
f(x,y) = a_1 y^2 + a_2 xy + a_3 x^2
\]

\[
= x^2 \left( a_1 \left(\frac{y}{x}\right)^2 + a_2 \frac{y}{x} + a_3 \right)
\]

Notice that \( x^2 \) is a hom. poly. of degree two and 

\[
a_1 \left(\frac{y}{x}\right)^2 + a_2 \left(\frac{y}{x}\right) + a_3
\]

is a hom. poly. of degree zero.

\[\text{Fact 3}\]

If \( f(x,y) \) is a hom. fun of degree \( n \) and \( g(x,y) \) is a hom. fun of degree \( m \) then \( f \cdot g \) is a hom. fun of degree \( m+n \).

Thus, we were able to rewrite \( f \) as a \( 2^{\text{nd}} \)-deg. poly. times a zero-degree poly.

Now consider the general poly of degree \( n \),

\[
f(x,y) = \sum_{k=0}^{n} a_k x^k y^{n-k} = y^n \sum_{k=0}^{n} a_k \frac{x^k}{y^{n-k}} = y^n f\left(\frac{x}{y}, 1\right),
\]

Equivalently, we can write the polynomial as

\[
f(x,y) = \sum_{k=0}^{n} a_k y^k x^{n-k} = x^n \sum_{k=0}^{n} a_k \left(\frac{y}{x}\right)^k = x^n f\left(1, \frac{y}{x}\right),
\]

where \( f\left(\frac{x}{y}, 1\right) \) and \( f\left(1, \frac{y}{x}\right) \) are both hom. fun. s of degree \( 0 \).

The reason we can write \( f \) in forms (1) and (2) is because each term is of the form \( x^k y^{n-k} = y^n \left(\frac{x}{y}\right)^k \) or \( y^k x^{n-k} = x^n \left(\frac{y}{x}\right)^k \).

It should be pointed out that forms (1) and (2) are true for any hom. fun. of deg=\( n \), not just polynomial fun. s (all \( \lambda \) \& \( y \)).\[ Note: \] If \( f \) is hom. with deg=\( n \) then 

\[
f(x,y) = x^n f\left(\lambda x, \lambda y\right).
\]

Let \( \lambda = \frac{1}{x} \) then 

\[
f(x,y) = x^n f\left(1, \frac{y}{x}\right) \text{ and } f(x,y) = y^n f\left(\frac{x}{y}, 1\right).
\]
Homogeneous ODEs

**Defn** An ODE of the form

\[ M(x, y) \, dx + N(x, y) \, dy = 0 \]  

is said to be homogeneous if both coefficients \( M \) and \( N \) are homogeneous functions of the same degree.

**Note** If we put (1) in the standard form

\[ M \, dx + N \, dy = 0 \]

\[
\begin{align*}
\frac{dx}{-M} & \quad \frac{dy}{N} = -M \\
\Rightarrow \quad \frac{dy}{dx} & = -\frac{M(x, y)}{N(x, y)} = f(x, y)
\end{align*}
\]

then \( f(x, y) \) is homogeneous of degree zero since

\[
\begin{align*}
f(\lambda x, \lambda y) & = -\frac{M(\lambda x, \lambda y)}{N(\lambda x, \lambda y)} = -\lambda^n \frac{M(x, y)}{\lambda^n N(x, y)} = -\frac{M}{N} = f(x, y).
\end{align*}
\]

Since \( f \) is homogeneous of degree 0 we can write it as

\[
\begin{align*}
f(x, y) & = f(1, \frac{y}{x}) \quad \text{or} \quad f(x, y) = f\left(\frac{x}{y}, 1\right).
\end{align*}
\]

The eqn can then be written as (we'll choose the 1st form for \( f \))

\[ \frac{dy}{dx} = f\left(1, \frac{y}{x}\right) \]  

(2)

Notice that the argument of \( f \) is \( \frac{y}{x} \). In some sense it is acting as a single variable. Thus the structure of eqn (2) suggests the substitution

**Let** \( u = \frac{y}{x} \) \( \Rightarrow \) \( y = ux \) \( \Rightarrow \) \( \frac{dy}{dx} = d(ux) = x\, du + u\, dx \) (product rule for differentials)

\[ \Rightarrow \frac{dy}{dx} = x \frac{du}{dx} + u \]  

(3)
Substituting (3) into (2) gives

\[ x \frac{du}{dx} + u = f(1, u) \]

\[ \rightarrow x \frac{du}{dx} = f(1, u) - u \]

\[ \rightarrow \frac{du}{f(1, u) - u} = \frac{dx}{x} \quad \text{(separation of variables)} \quad (4) \]

\[ \text{back sub.:} \quad \frac{du}{-M(1, u) - u} = \frac{dx}{x} \]

\[ \text{Back substitute for } f = -\frac{M}{N} \]

\[ f(x, y) = -\frac{M(x, y)}{N(x, y)} \rightarrow f(1, u) = \frac{-x^n M(1, u)}{X^n N(1, u)} \]

\[ \text{where } u = \frac{y}{x} \]

\[ \rightarrow \frac{-N(1, u) \, du}{M(1, u) + u \, N(1, u)} = \frac{dx}{x} \quad (5) \]

Thus if \( M \) and \( N \) are homogenous functions then we can reduce a homogenous ODE to a separable eqn by a substitution.

**Note:** Since \( f(x, y) = f\left(\frac{x}{y}, 1\right) \) we could have made the substitution \( v = \frac{x}{y} \). Both methods are equivalent. However, this does not mean that both methods are equally easy to apply in practice.

A good rule of thumb for \( M \, dx + N \, dy = 0 \) if \( M \) is of a simpler form than \( N \), then substitute for \( x \) \( (v = \frac{x}{y} \text{ or } x = vy) \) and if \( N \) is simpler, sub. for \( y \) \( (y = ux) \).

**EX** For the ODE \( xy \, dx + (x^2 + y^2) \, dy = 0 \) let \( x = vy \). Since \( M = xy \) is of a simpler form that \( N = x^2 + y^2 \).
Example 1 [Homogeneous functions]
Determine whether the fun. s are homogeneous. If so, state degree.

(i) \( f(x,y) = \sqrt{x+y} (ax+by) \)
\[ f(\lambda x, \lambda y) = \sqrt{\lambda x + \lambda y} (a \lambda x + b \lambda y) = \lambda^{\frac{3}{2}} f(x,y) \]
homo., deg = \( \frac{3}{2} \)

(ii) \( f(x,y) = \frac{3x}{ax^2 + bx + cy^2} \)
homo, deg = -1

(iii) \( f(x,y) = \tan \left( \frac{x-y}{x+y} \right) \)
homo, deg = 0

(iv) \( f = \frac{-lnx^3}{lny^3} = \frac{lnx}{lny} \neq ln \left( \frac{x}{y} \right) \) not homo, can't bring \( \lambda \) out of the argument of \( ln x, \ ln y \).

(v) \( f = (x+y+1)^2 \) not homo, cu \( x+y+1 \) is not homo.

Example 2 Solve the homogeneous eqn

\((x+y)dx + xdy = 0.\)

Solution

Let \( y = ux \) since \( N = x \) is simpler than \( M = x+y \). Then

\( dy = xdu + udx \rightarrow \)

\[(x+ux)dx + x(xdu + udx) = 0 \rightarrow \frac{dx}{1+2u} + \frac{du}{1+2u} = 0 \rightarrow \ln |x(1+2u)| = \ln |c| \]

\[ \rightarrow x(1+2u) = c \rightarrow u = \frac{1}{2} \left( \frac{c^2}{x^2} - 1 \right) \]

\[ \rightarrow \frac{y}{x} = \frac{1}{2} \left( \frac{c^2}{x^2} - 1 \right) \rightarrow y = \frac{1}{2} \left( \frac{c^2 - x^2}{x} \right) \rightarrow 2xy + x^2 = c^2 = c_1 \]

Example 3

\[ x \frac{dy}{dx} - y = \sqrt{x^2 + y^2} \rightarrow xdy = \left( y + \sqrt{x^2 + y^2} \right) dx \]

\( x(udx + xdu) = (ux + \sqrt{x^2 + u^2})dx \) (Assume \( x > 0 \) so \( \sqrt{x^2 + u^2} = x\sqrt{1+u^2} \))

\[ \frac{dx}{x} = \frac{du}{\sqrt{1+u^2}} \rightarrow \ln |x| = \ln |u + \sqrt{1+u^2}| + \ln c \]

\[ \rightarrow x = c(u + \sqrt{1+u^2}) = c \left( \frac{y}{x} + \sqrt{1+\left( \frac{y}{x} \right)^2} \right) = \frac{c}{x} \left( y + \sqrt{x^2 + y^2} \right) \]

\[ \rightarrow x^2 = c \left( y + \sqrt{x^2 + y^2} \right) \]
Example 5
\[
\frac{dy}{dx} = \frac{y}{x} \ln \left(\frac{y}{x}\right).
\]
Let \( u = \frac{y}{x} \), \( \frac{dy}{dx} = u + x \frac{du}{dx} \). Then \( eq \) becomes
\[
u + x \frac{du}{dx} = u \ln u \implies x \frac{du}{dx} = u (\ln u - 1) \implies \frac{du}{u (\ln u - 1)} = \frac{dx}{x}
\]
\[
\int \frac{1}{\ln u - 1} \frac{du}{u} = \int \frac{dx}{x} \quad \text{(Note:} \quad \frac{du}{u} = d(\ln u) \text{ so let } w = \ln u - 1 \quad \text{)}
\]
\[
\int \frac{dw}{w} = \int \frac{dx}{x} \quad \implies \quad |\ln w| = |\ln x| + |\ln c| = \ln (cx)
\]
\[
e_1 \quad w = cx \quad \text{back sub} \quad \implies \quad \ln u - 1 = \ln \left(\frac{y}{x}\right) - 1 = cx \implies \ln \left(\frac{y}{x}\right) = cx + 1
\]
\[
\implies y = x e^{cx + 1}
\]

\[
y \ dx + \left( y \cos \left( \frac{x}{y} \right) - x \right) \ dy = 0 \quad \overset{\text{cancel}}{\implies} \quad dx + \left( \cos \left( \frac{x}{y} \right) - \frac{x}{y} \right) \ dy = 0.
\]
The form of the \( eq \) and the argument of cosine suggests the substitution \( v = \frac{x}{y} \) \( (x = vy) \), \( dx = v \ dy + y \ dv \). The \( eq \) becomes
\[
v \ dy + y \ dv + \left( \cos v - v \right) \ dy = 0 \implies y \ dv + \cos v \ dv = 0
\]
\[
s \quad \int |\sec v + \tan v| + |\ln y| = \ln c
\]
\[
e_2 \quad y (\sec v + \tan v) = c \quad \implies \quad y \left( \sec \left( \frac{x}{y} \right) + \tan \left( \frac{x}{y} \right) \right) = c
\]
\[
(\sqrt{x} + \sqrt{y})^2 \ dx = x \ dy \quad (y(1) = 0) \quad \text{(Note:} \quad x > 0 \text{ here,} \quad \text{So} \quad \sqrt{x^2} = x \quad \text{)}
\]
\[
\frac{x^2}{2} (1 + \sqrt{x})^2 \ dx = x \ dy \quad \implies \quad x \left( 1 + \sqrt{x} \right)^2 \ dx = x \ dy \overset{\text{sub}}{\implies} \left( 1 + \sqrt{x} \right)^2 \ dx = dy
\]
Let \( u = \frac{y}{x} \), \( dy = u \ dx + x \ du \implies \frac{du}{dx} = \frac{u + x \ du}{dx} \). The \( eq \) becomes
\[
\frac{dy}{dx} = \left( 1 + \sqrt{x} \right)^2 \implies u + x \frac{du}{dx} = \left( 1 + \sqrt{u} \right)^2 = 1 + 2 \sqrt{u} + u
\]
\[
\implies x \frac{du}{dx} = 1 + 2 \sqrt{u} \implies \frac{du}{1 + 2 \sqrt{u}} = \frac{dx}{x} \implies \int \frac{dw}{1 + 2 \sqrt{u}} = \int \frac{1}{2} \left( \frac{w-1}{w} \right) \ dw = \frac{1}{2} \int \left( 1 - \frac{1}{w} \right) \ dw = \frac{1}{2} (w - \ln w) + c = \frac{1}{2} (1 + 2 \sqrt{u} - \ln (1 + 2 \sqrt{u})) + c
\]
\[
\text{Finish} \quad \frac{w-1}{w} \ dw = \frac{u^{1/2} \ du}{u^{1/2}} \Rightarrow \quad du = u^{1/2} \ dw = \frac{1}{2} (w-1) \ dw
\]