Solving linear $1^{st}$-order Eqn.'s

The general form of a $1^{st}$ order linear eqn is

$$a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \quad (1)$$

Dividing (1) by the leading coeff. $a_1(x)$ will put the eqn in the standard form

$$\frac{1}{a_1(x)} \frac{dy}{dx} + \frac{a_0(x)}{a_1(x)} y = \frac{g(x)}{a_1(x)}$$

re-label

$$\frac{dy}{dx} + P(x) y = f(x) \quad \text{(Standard Form)} \quad (2)$$

We seek a sol'n on the largest interval $I$ where both $P(x)$ and $f(x)$ are continuous. We cannot just integrate (2) as it stands. But what if $\frac{dy}{dx} + P(x) y = \frac{d}{dx} (H(x) y)$, then we could solve

solve (2) by integrating both sides:

$$\frac{d}{dx} (H(x) y) = f(x) \quad \rightarrow \quad H(x) y = \int f(x) dx \quad \rightarrow \quad y(x) = \frac{\int f(x) dx}{H(x)}.$$

However, $\frac{dy}{dx} + P(x) y \neq \frac{d}{dx} (H(x) y) = H(x) \frac{dy}{dx} + H'(x) y$.

But we're not sunk yet!

What if we tried multiplying $\frac{dy}{dx} + P(x) y$ by a fcn $\mu(x)$, called an integrating factor, that would convert $\frac{dy}{dx} + P(x) y$ into a perfect differential?

$$\mu \rightarrow \mu \frac{dy}{dx} + \mu P(x) y = H(x) \frac{dy}{dx} + H'(x) y$$

Equating coeffs:

$$\mu = H \quad \text{and} \quad H' = \mu P = P(x) H \quad (3)$$
We can solve for the unknown $H(x)$ (or equivalently, $\mu$) in (3) using separation of variables.

\[ \frac{d\mu}{dx} = P(x) \mu \]

Separate
\[ \frac{d\mu}{\mu} = P(x) \]

Integrating factor (I.F.)
\[ e^{\int P(x) dx} \]

Multiplying eqn (2) through by $\mu$ yields
\[ \mu \cdot (2) \]

LHS
\[ \frac{d}{dx} \left[ e^{\int P(x) dx} y \right] = e^{\int P(x) dx} f(x) \]

By construction, the LHS of the previous eqn is a perfect differential
\[ e^{\int P(x) dx} y = \int e^{\int P(x) dx} f(x) dx + C \]

Eqn (5) is the general solution since it contains one arbitrary constant of integration.
Comment: You should not solve problems in practice using formula (5), instead you should follow the procedure used to derive formula (5).

Outline of procedure:

Step 1: Put ODE in standard form: \( y' + P(x)y = f(x) \).

Step 2: After determining \( P(x) \), use it to generate the integrating factor \( \mu(x) \) via

\[
\mu(x) = e^{\int P(x) \, dx}
\]

(Note) The exponent should be a function of \( x \) with no arbitrary constant of integration in it.

Step 3: Multiply the ODE by the I.F. \( \mu(x) \) that you computed in step 2 and use the fact that the LHS is now a perfect diff.

\[
e^{\int P(x) \, dx} y' + e^{\int P(x) \, dx} P(x)y = e^{\int P(x) \, dx} f(x)
\]

\[
\rightarrow \quad \frac{d}{dx} \left[ e^{\int P(x) \, dx} y \right] = e^{\int P(x) \, dx} f(x) \quad \text{(by construction the LHS of the eqn is a perfect diff.)}
\]

Step 4: Integrate both sides of eqn found in step 3 and solve for \( y \).
Instructions: Find the general soln to the following linear eqns. If any I.C. is given, then solve for the unique soln.

Example 1

**Linear 1st-order ODEs**

**Solution:**

\[ \frac{dy}{dx} + 2y = 3 \]  \( \text{(This is an equidimensional eqn)} \)

\[ \text{(Could solve this using separation of variables.)} \]

**Step 1**

\[ \frac{dy}{dx} + \frac{2}{x} y = \frac{3}{x} \] (1)  \( \text{(Put in standard form: } P = \frac{2}{x}, \quad f = \frac{3}{x}) \)

**Step 2**

Find I.F.: \( \mu(x) = e^{\int \frac{2}{x} \, dx} = e^{2 \ln x} = e^{\ln x^2} = x^2 \).

**Note:** We did not add an arbitrary constant to the integrating factor.

**Step 3**

\[ x^2 \frac{dy}{dx} + 2xy = 3x \]

\[ \frac{d}{dx} (x^2 y) = 3x \]

\[ \int x^2 y \, dx = \int 3x \, dx + c = \frac{3}{2} x^2 + c \]

\[ y = \frac{3}{2} x + \frac{c}{x^2} \]

Example 2

Solve: \( (1 + x) \frac{dy}{dx} - xy = x(x+1) \)

**Step 1**

\[ \frac{dy}{dx} - \frac{x}{x+1} y = x \] (1)  \( \text{(Standard form: } P(x) = \frac{-x}{x+1}, \quad f(x) = x) \)

**Step 2**

Compute I.F.: \( \int \frac{x}{x+1} \, dx = \int \frac{u-1}{u} \, du = u - \ln u = x + 1 - \ln(x+1) \)

let \( u = x+1 \), \( du = dx \) (for \( x > -1 \))

\[ \mu(x) = e^{-\int \frac{x}{x+1} \, dx} = e^{-(x+1) \ln (x+1)} = e^{-(x+1)} \cdot e^{\ln(x+1)} = (x+1)e^{-x} \)

**Step 3**

\[ (x+1)e^{-x} \frac{dy}{dx} - x e^{-x} y = x(x+1)e^{-x} \]  \( \text{(This simplifies the eqn a little bit.)} \)

\[ \text{Getting rid of unnecessary const.} \]

LHS is exact der:

\[ \frac{d}{dx} [(x+1)e^{-x} y] = (x^2 + x)e^{-x} \] (Z)
**Example 3**

Solve \( y' = \frac{3e^y - 2x}{3} \). Notice that the ODE is non-linear in \( y \). However, if we invert the dependency from \( y \) as a function of \( x \) to \( x \) as a function of \( y \) (\( x = x(y) \), \( \frac{dx}{dy} \)) then the eqn is linear in \( x \).

\[
\frac{dx}{dy} = (3e^y - 2x) dy
\]

\[
\frac{dx}{dy} = 3e^y - 2x
\]

\[
\frac{dx}{dy} + 2x = 3e^y
\]

Let \( u(y) = e^{\int 2dy} = e^{2y} \) be the integrating factor.

\[
e^{2y} \frac{dx}{dy} + e^{2y} 2x = e^{2y} (3e^y) = 3e^{3y}
\]

\[
\frac{d}{dy} \left[ e^{2y} x \right] = 3e^{3y} = \frac{d}{dy} (e^{3y})
\]

\[
\int dy \quad e^{2y} x = e^{3y} + c
\]

\[
\therefore e^{-2y} x = e^y + c e^{-2y}
\]

\[
\therefore x(y) = e^y + c e^{-2y}
\]

This is a good problem because it demonstrates that sometimes it may be very difficult or impossible to solve an ODE with independent variable \( x \), dependent variable \( y \), but by exchanging the roles of independent and dependent variables, the eqn can be solved.
Example 4: Show that all first-order linear homogeneous eqns.
can be solved using the method of separation of variables.

\[ \frac{dy}{dx} + P(x)y = Q(x) = 0 \Rightarrow \frac{dy}{y} = -P(x)dx. \]
\[ \int \ln |y| = \int P(x)dx + c, \quad e^{\ln y} = c \cdot e^{P(x)dx}. \]

Example 5: \( y \frac{dx}{dy} - x = 2y^2 \) with I.C. \( y(1) = 5 \). Solve for \( x \) as a function of \( y \).

Solution:

Step 1: Put in standard form: \( \frac{dx}{dy} + P(y)x = f(y) \).
\[ \frac{dx}{dy} - \frac{1}{y} x = 2y \quad (P(y) = -\frac{1}{y}, \quad f(y) = 2y) \] (1)

Step 2: Compute I.F.
\[ u = e^{-\int P(y)dy} = e^{-\ln y} = e^{\ln \frac{1}{y}} = \frac{1}{y} \]

\[ \left. \frac{1}{y} \right|_{(1)} \frac{dx}{dy} - \frac{1}{y^2} x = 2 \]
\[ \frac{d}{dy} \left( \frac{1}{y} x \right) = 2 \]
\[ \int dy = \frac{1}{y} x = 2y + c \]

\[ \rightarrow X(y) = 2y^2 + cy \] (2)

To determine the constant in eq 2, we apply the I.C.
\( y(1) = 5 \Rightarrow x_0 = 1 \) and \( y_0 = 5 \) so the solution curve passes through the point \((1, 5)\) \( \Rightarrow \) the I.C. in \( X(y_0) = x_0 \) becomes \( X(5) = 1 \). Applying this to eq 2 yields

\[ X(5) = 2 \cdot 5^2 + c \cdot 5 = 1 \]
\[ \Rightarrow 10 + c = \frac{1}{5} \rightarrow c = \frac{1}{5} - 10 = \frac{-49}{5} \] (3)

Substituting \( c = \frac{-49}{5} \) into (2) gives the unique solution curve
\[ X(y) = 2y^2 - \frac{49}{5}y \]
Example 6 Find a continuous soln to the ODE with a discontinuous forcing term: \( \frac{dy}{dx} + y = f(x) \), \( f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 & x > 1 \end{cases} \) with I.C. \( y(0) = 1 \).

The eqn is already in standard form.

Notice that \( f(x) \) is discontinuous at \( x = 1 \).

We seek a cont. soln.

**Step 1** Solve the ODE on each region: \( 0 \leq x \leq 1 \) and \( x > 1 \).

On \( [0, 1] \), \( f(x) = 1 \) and the eqn becomes

\[
\frac{dy}{dx} + y = 1 \quad (P(x) = f(x) = 1), \text{ with I.C. } y(0) = 1 \quad (1)
\]

\[ M = e^{ \int 1 \, dx } = e^x \]

\[ M \cdot (1) \rightarrow e^x \frac{dy}{dx} + e^x y = e^x \rightarrow \frac{d}{dx}(e^x y) = e^x \]

\[ \int e^x \, dx \rightarrow e^x y = e^x + c \]

\[ y = 1 + ce^{-x} \quad \text{on } [0, 1] \quad (2) \]

Since the I.C. is given at \( x = 1 \) and \( 1 \) is contained in the interval \([0, 1]\) we must apply the I.C. to the soln (2).

\[ 1 = y(0) = 1 + ce^{0} = 1 + c \quad \Rightarrow \boxed{c = 0}. \quad (3) \]

Thus, \( y = 1 \) is the soln to the ODE on \([0, 1]\). \( (4) \)

Next, we solve the ODE on the interval \( x > 1 \) \((1, \infty)\). On \( x > 1 \), \( f(x) = -1 \) and the ODE becomes

\[
\frac{dy}{dx} + y = -1 \quad (P(x) = 1, \ f(x) = -1).
\]

Since \( P(x) = 1 \) is the same as the \( P \) corresponding to the interval \([0, 1]\), we will have the same I.F. \( M = e^x \). The eqn becomes

\[ e^x \frac{dy}{dx} + e^x y = -e^x \rightarrow \frac{d}{dx}(e^x y) = -e^x \rightarrow e^x y = -e^x + c \]

\[ \Rightarrow \boxed{y = -1 + ce^{-x}} \quad \text{on } x > 1 \quad (5). \]

For \( y \) to be cont. on \([0, \infty)\) demand \( y(1) = y(1) \Rightarrow -1 + ce^{-1} = 1 \Rightarrow c = 2e \Rightarrow \boxed{y(x) = -1 + 2e^{1-x}} \)