Additional Tricks: Combining exact differentials

Sometimes we can integrate a differential eqn just by recognizing it, or a sub component, as an exact differential. You should memorize the following:

\[ d(xy) = x\,dy + y\,dx \quad (1) \]
\[ d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2} \quad (2) \]
\[ d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2} \quad (3) \]
\[ d\left(\tan^{-1}x\right) = \frac{dx}{1+x^2} \quad (4) \]

\[ d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot d\left(\frac{y}{x}\right) \quad \text{(Using chain rule)} \]

\[ = \frac{1}{1 + \frac{y^2}{x^2}} \left[ \frac{x\,dy - y\,dx}{x^2} \right] \quad \text{(Using \ 3)} \]

\[ = \frac{x\,dy - y\,dx}{x^2 + y^2} \]

\[ d\left(e^{\int p(x)\,dx} \cdot y\right) = \left(y' + p(x)\,y\right)\,e^{\int p(x)\,dx} \]
Combining exact differentials

Example

\[ y \, dx + (x + x^3 y^2) \, dy = 0 \]

Regrouping terms of like degree

\[ (x \, dy + y \, dx) + \frac{x^3 y^3}{y} \, dy = 0 \]

\[ \frac{d(xy)}{(xy)^3} + \frac{dy}{y} = 0 \]

\[ \frac{-1}{2} d\left[(xy)^2\right] + d(\ln|y|) = 0 \quad \text{(2 exact differentials)} \]

\[ d\left[\ln|y| - \frac{1}{2} (xy)^{-2}\right] = 0 \quad \text{(we've reduced it to)} \]

\[ \frac{\int d}{\int} \ln|y| - \frac{1}{2} (xy)^{-2} = -\ln|c| \]

\[ \ln|cy| = \frac{1}{2(\ln|y|)^2} \quad \rightarrow \quad (xy)^2|\ln|cy|| = 1 \]
Combining Exact Differentials

\[ \text{Ex. 2} \] Solve \( y (x^3 - y) \, dx - x (x^3 + y) \, dy = 0 \).

Start by expanding terms:

\[ x^3 \, dx - y^2 \, dx - x^4 \, dy - xy \, dy = 0 \]

Combine like powered terms and factor out any common factors:

\[ -y [xdy + ydx] + x^3 [ydx - xdy] = 0 \]

\[ \frac{\dot{y}^2}{y} - \frac{d(xy)}{y} + x^3 \left[ \frac{ydx - xdy}{y^2} \right] = 0 \]

\[ - \frac{d(xy)}{y} - x^3 \, d \left( \frac{y}{x} \right) = 0 \]

\[ \frac{\dot{x}}{xy} \frac{d(xy)}{y} - x^2 \, d \left( \frac{y}{x} \right) = 0 \]  \( (1) \)

We need to make \((1)\) an exact differential. To do this we will need to multiply by an integrating factor \( \mu(x,y) \). It should be pointed out that in this case \( \mu \neq e^{\int \frac{dy}{x}} \) because \((1)\) is not in the standard form of a linear eq^n. In fact, \((1)\) is not a linear eq^n.

We try an integrating factor of the form: \( \mu = \frac{1}{(xy)^n} \), since in order to keep the first term a perfect differential \( x \) and \( y \) must occur in the form of \( xy \). The hope is that there is a value of \( n \) for which the \( 2^{nd} \) term will be of the form \((xy)^m\).

\[ \mu \cdot (1) \rightarrow \frac{d(xy)}{(xy)^{n+1}} - \frac{x^2}{(xy)^n} \, d \left( \frac{y}{x} \right) = 0 \]  \( (2) \)
Combining Exact Differentials

We want the 2nd term to be of the form

\[ \frac{x^2}{x^n y^n} = \frac{x^{2-n}}{y^n} = \left( \frac{x}{y} \right)^m \]

\[ \implies \text{Demand } x^{2-n} = x^m \text{ and } y^n = y^m \]

The condition \[ y^n = y^m \] \[ \implies n = m \]. The condition \[ x^{2-n} = x^m \] \[ \implies 2-n = m = n \] \[ \implies n = 1 \].

\[ n = 1 \] is the only solution.

Thus, our I.F. is \[ u(x, y) = \frac{1}{xy} \].

Setting \[ n = 1 \] in eqn. (2) yields a perfect derivative,

\[ \frac{d(xy)}{(xy)^2} - \frac{x}{y} \frac{d(x)}{(x)^2} = 0 \]

\[ \implies (xy)^{-2} d(xy) - \frac{1}{x} d(x) = 0 \] \[ \text{(4)} \]

To see that both terms in (4) are perfect der. s let \[ w = xy \] and \[ u = \frac{y}{x} \] then (4) becomes

\[ w^{-2} dw - \frac{1}{u} du = 0 \]

\[ \implies d(w^{-1}) - d(\ln|u|) = 0 \]

\[ \implies d[w^{-1} - \ln|u|] = 0 \]

\[ \implies d[(xy)^{-1} - \ln|\frac{y}{x}|] = 0 \]

\[ \therefore \int \frac{1}{xy} - \ln|\frac{y}{x}| = C \]
Combining Exact differentials 

**Example 3** Sometimes it is helpful to make a substitution before you examine an eqn as shown in our last example.

Solve

\[(1 + 3x^2 \cos y) \, dx - x^3 \sin y \, dy = 0. \tag{1}\]

**Solution** This is a very nonlinear eqn and would be nontrivial to integrate directly.

**Step 1** Notice that one term contains a \( \cos y \) and the other a \( \sin y \).

This suggests the substitution:

Let \( u = \cos y \rightarrow du = -\sin y \, dy \). Then the ODE in (1) becomes

\[(1 + 3x^2 u) \, dx + x^3 du = 0\]

Grouping terms of the same order and noticing that \( 3x^2 \, dx = d(x^3) \)

\[dx + [d(x^3)u + x^3 \, du] = 0\]

\[\rightarrow dx + d(x^3 u) = 0\]

\[\rightarrow d(x + u x^3) = 0 \quad \text{(combine terms)}\]

\[\rightarrow d(x + x^3 \cos y) \quad \text{(back substitute for } u)\]

\[\int x + x^3 \cos y \, dy = \text{constant}\]
Consider the general 1st order ODE:

$$M(x,y) \, dx + N(x,y) \, dy = 0. \quad (1)$$

This need not be a linear eqn. In section 2.5 we found the integrating factor for a linear ODE. We now try to find an IF, for (1). Let $\mu(x,y)$ be our integrating factor. Then upon multiplying $\mu$ by (1) we have

$$\mu \cdot (eq \,(1)) \quad (\mu M) \, dx + (\mu N) \, dy = 0. \quad (2)$$

In order for this eqn to be integrable, the eqn must be exact. That is, the eqn is integrable if and only if $\exists f :$

$$(\mu M) \, dx + (\mu N) \, dy = f_x \, dx + f_y \, dy = df$$

then

$$0 = (\mu M) \, dx + (\mu N) \, dy = df \implies f(x,y) = \text{constant}$$

and we have our solution curve.

In other words, $\mu$ must make eqn (2) exact in order for it to be an integrating factor. Thus, we require

$$\partial_y (\mu M) = \partial_x (\mu N)$$

$$\iff \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\iff \mu (M_y - N_x) = \mu_x N - \mu_y M. \quad (3)$$

Eqn (3) is a partial differential eqn, with coefficients $M$ and $N.$

Note: $M$ and $N$ are known functions.

The approach we have taken is a very natural one. Unfortunately, it has produced a partial, not ordinary, differential eqn. We are stuck! We will need to make a simplifying assumption to proceed any further.
Integrating factors: the general case for 1st order ODEs

Let us rewrite eqn (3) into a more useful form:

\[
\frac{\dot{\mu}}{\mu} = M_y - N_x = \frac{\mu_x}{\mu} N - \frac{\mu_y}{\mu} M \quad (4)
\]

Next, let's multiply (4) by \( \frac{dx}{N} \):

\[
\frac{1}{N} (M_y - N_x) \, dx = \frac{\mu_x}{\mu} \, dx - \frac{\mu_y}{\mu} \frac{M}{N} \, dx \quad (5)
\]

If we now assume that \( \frac{1}{N} (M_y - N_x) \) is a function of \( x \) alone, then by restricting our search to \( \mu = \mu(x) \) we can reduce the PDE in (5) to an ODE since \( \mu_y = 0 \) in this case. (Note: \( \mu = \mu(x) \) is a particular sol\( \) of the PDE)

Case I: \( \partial_y \left[ \frac{1}{N} (M_y - N_x) \right] = 0 \) and \( \mu_y = 0 \).

Eqn (5) becomes

\[
\frac{d\mu}{\mu} = d \ln \mu = \frac{1}{N} (M_y - N_x) \, dx \quad \text{(where } \frac{\mu_x}{\mu} \, dx = d\mu \text{ since } \mu \text{ is a func of } x \text{ alone)}
\]

\[
\int \ln \mu = \int \frac{1}{N} (M_y - N_x) \, dx \quad \text{(Note: We do not include an arbitrary constant since we only seek a particular soln.)}
\]

\[
e^c \quad \mu = e^{\int \frac{1}{N} (M_y - N_x) \, dx} \quad \text{(we can drop absolute value on } \mu \text{ since } e^z > 0 \text{ for any real number } z)
\]

Suppose that

\[
\partial_y \left[ \frac{1}{N} (M_y - N_x) \right] \neq 0 \quad \text{that is } \frac{1}{N} (M_y - N_x) \text{ is not a}
\]

function of \( x \) alone. Then, we still have one more hope, we must check to see if \( \frac{1}{M} (M_y - N_x) \) is a function of \( y \) alone.
Integrating factors: the general case for 1st order ODEs

Case 2

We now seek a solution similar to (6). Multiply eqn (4) by

\[-\frac{dy}{M} = \frac{1}{M} (M_y - N_x) \, dy = -\frac{N}{M} \, dy + \frac{M}{\mu} \, dy\]  

(7)

If we assume that \(-\frac{1}{M} (M_y - N_x)\) is a function of \(y\) alone, and we seek a particular solution to the PDE in (7) of the form \(\mu = \mu(y)\) (a function of \(y\) alone) then eqn (7) becomes the ODE

\[-\frac{1}{M} (M_y - N_x) \, dy = \frac{\mu y}{\mu} \, dy = \frac{d\mu}{\mu} = d \ln |\mu|\]

\[
\int \ln |\mu| = -\int \frac{1}{M} (M_y - N_x) \, dy
\]

(8)

Since we only seek a particular solution we don't add an arbitrary constant.

\[
e^{(\cdot)} \mu = e^{-\int \frac{1}{M} (M_y - N_x) \, dy}
\]

If \(-\frac{1}{N} (M_y - N_x)\) was not a function of \(x\) alone, that is \(\exists y \left[ \frac{1}{N} (M_y - N_x) \right] \neq 0\) and \(-\frac{1}{M} (M_y - N_x)\) was not a function of \(y\) alone \(\exists x \left[ \frac{\mu}{M} (M_y - N_x) \right] \neq 0\) then eqn's (6) and (8) would not be integrating factors. In this case, we would be forced to confront the partial differential eqn found in (4) to find an integrating factor that depends on both \(x\) and \(y\). However, such an endeavor will not be taken up here!
We now summarize what we have found.

Case (i) If \( \frac{1}{N} (\partial_y M - \partial_x N) \) is a function of \( x \) alone,
then \( \exists \) an integrating factor \( \mu = \mu(x) \) for the eqn
\[ M \, dx + N \, dy = 0. \]
Moreover, \( \mu \) is given by
\[ \mu(x) = e^{\int \frac{1}{N} (\partial_y M - \partial_x N) \, dx}. \]

Case (ii) If \( \frac{1}{M} (\partial_x M - \partial_y N) \) is a function of \( y \) alone,
then \( \exists \) an integrating factor \( \mu = \mu(y) \) for the eqn
\[ M \, dx + N \, dy = 0. \]
Moreover, \( \mu \) is given by
\[ \mu(y) = e^{-\int \frac{1}{M} (\partial_x M - \partial_y N) \, dy}. \]
Integrating factors for 1st order nonlinear ODE

Solve \[ y(x+y+1) \, dx + x(x+3y+2) \, dy = 0 \]

Let \[
\begin{align*}
M &= y(x+y+1) = xy + y^2 + y \\
M_y &= x + 2y + 1 \\
N &= x(x+3y+2) = x^2 + 3xy + 2x \\
N_x &= 2x + 3y + 2
\end{align*}
\]

Form \( M - N_x \). The structure of this term will determine if we divide by \( M \) or \( N \). It may also be the case that neither dividing by \( M \) nor \( N \) will produce a function of \( x \) or \( y \) alone.

\[
M_y - N_x = x + 2y + 1 - (2x + 3y + 2)
= -x - y - 1 = -(x+y+1) = -\frac{M}{y}
\]

\[
\Rightarrow \frac{-1}{M} (M_y - N_x) = \frac{1}{y} \quad \text{(a function of } y \text{ alone)}
\]

\[
\Rightarrow \mu = e^{\int \frac{-1}{M} (M_y - N_x) \, dy} = e^{\int \frac{1}{y} \, dy} = e^{\ln|y|} = |y|
\]

Assume \( y > 0 \). Then \( \mu = y \) is the integrating factor for the ODE. Thus \( \mu \cdot \text{eqn}(1) \) becomes an exact eqn.

\[
\mu \cdot \text{eqn}(1) \rightarrow y^2(x+y+1) \, dx + xy(x+3y+2) \, dy = 0 \quad \text{(exact eqn)}
\]

\[
\Rightarrow y^2dx + y^3dx + x^2ydy + x3y^2dy + x2ydy = 0
\]

\[
\Rightarrow \frac{1}{2} \, y^2d(x^2) + y^3dx + x^2d(y^2) + \frac{1}{2} \, x^2d(y^2) + xd(y^3) + xd(y^2) = 0
\]

\[
\Rightarrow \frac{1}{2} \, [y^2d(x^2) + x^2d(y^2)] + [y^3dx + xd(y^3)] + [y^2dx + xd(y^2)] = 0
\]

\[
\Rightarrow \frac{1}{2} \, d(x^2y^2) + d(xy^3) + d(xy^2) = 0
\]

\[
\Rightarrow \int \frac{1}{2} \, x^2y^2 + xy^3 + xy^2 = c_1 \Rightarrow xy^2(x+2y+2) = c
\]
Integrating factors for 1st order nonlinear ODEs

\[ (x^2 + y^2 + 1) \, dx + x(x-2y) \, dy = 0. \]  \hspace{1cm} (1)

**Ex. 2** Solve the eqn

\[ (x^2 + y^2 + 1) \, dx + x(x-2y) \, dy = 0. \]

\[ M = x^2 + y^2 + 1 \quad \text{and} \quad N = x(x-2y) = x^2 - 2xy \]

\[ M_y = 2y, \quad N_x = 2x - 2y = 2(x-2y) \]

First compute \( M_y - N_x \) to determine if we divide by \( M \) or \( N \).

\[ M_y - N_x = 2y - (2x - 2y) = 4y - 2x = -2(x-2y) \Rightarrow \text{divide by } N \]

Then

\[ \frac{1}{N} (M_y - N_x) = \frac{4y - 2x}{x(x-2y)} = \frac{-2(x-2y)}{x(x-2y)} = \frac{-2}{x} \]

Thus

\[ \int \frac{1}{N} (M_y - N_x) \, dx = \int \frac{-2}{x} \, dx = -2 \ln |x| = -2 \]

Multiplying (1) by \( \mu \) yields an exact eqn (by construction)

\[ \frac{x^2 + y^2 + 1}{x^2} \, dx + \frac{x-2y}{x} \, dy = 0 \]

\[ \rightarrow \left(1 + \frac{y^2}{x^2} + x^{-2}\right) \, dx + \left(1 - 2y \, x^{-1}\right) \, dy = 0 \]

\[ \rightarrow dx + \frac{y^2}{x^2} \, dx - d(x^{-1}) + dy - \frac{2y \, dy}{x} \, x^{-1} = 0 \]

\[ \rightarrow \begin{align*}
&d(x - y^2 \, d(x^{-1}) - d(x^{-1}) + dy - \frac{d(y^2) \, x^{-1}}{d(y^2)} \\
&\text{group as a product rule: } - \left[y^2 \, d(x^{-1}) + d(y^2) \, x^{-1}\right]
\end{align*} \]

\[ \rightarrow d(x + y - x^{-1}) = d(y^2 \, x^{-1}) = 0 \]

\[ \rightarrow x + y - \frac{1}{x} - \frac{y^2}{x} = c \]

\[ \Rightarrow x^2 - y^2 + xy - 1 = cx \]