Sometimes we can reduce an eqn to a Bernoulli eqn.

\[ \frac{dy}{dx} = P(x) + Q(x) \cdot y + R(x) \cdot y^2 \quad (1) \]

is called a Ricatti Eqn.

In general, these eqn's are very difficult to solve. The reason for this is that in order to reduce a Ricatti eqn to a Bernoulli eqn, you must first know a particular soln! And there is no procedure for how to get this soln. This means that you must guess a soln!

\[ \text{Note: If } P, Q, \text{ and } R \text{ are all polynomials then try looking for a soln of the form } y = \text{polynomial in } x \text{. (Try } y = ax^c \text{ first.)} \]

Suppose you know a particular soln \( y = y_p(x) \). Then let

\[ y = u(x) + y_p(x) \]

and

\[ \frac{dy}{dx} = \frac{du}{dx} + \frac{dy_p}{dx} \]

in eqn (1) to get

\[ \frac{du}{dx} + \frac{dy_p}{dx} = P(x) + Q(x) \cdot (u + y_p) + R(x) \cdot (u + y_p)^2 \]

\[ = P(x) + Q(x) \cdot u + Q(x) \cdot y_p + R(x) \cdot (u^2 + 2uy_p + y_p^2) \]

\[ = P(x) + Q(x) \cdot y_p + R(x) \cdot y_p^2 + Q(x) \cdot u + R(x) \cdot (u^2 + 2uy_p) \]

\[ \frac{dy_p}{dx} = \left( \frac{dy_p}{dx} \right) \text{ (by hypothesis)} \]

\[ \rightarrow \frac{du}{dx} = (Q + 2y_p \cdot R) \cdot u = R(x) \cdot u^2 \quad \text{(Bernoulli Eqn)} \]

\[ \rightarrow \frac{du^2}{dx} = (Q - 2y_p \cdot R) \cdot u^2 = R(x) \quad \text{(Let } w = u^2, \text{ dw} = -u^2 \, \text{du) } \]

\[ \rightarrow -\frac{dw}{dx} = (Q - 2y_p \cdot R) \cdot w = R(x) \quad \text{ (1)} \]

\[ \rightarrow \frac{d}{dx} \left( e^{\int (Q - 2y_p \cdot R) \, dx} \cdot w \right) = -Re^{\int (Q - 2y_p \cdot R) \, dx} \quad \text{Finish} \]
Converting a Ricatti eqn to a 2nd order linear ODE.

\[
\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2
\]  

(1)

**Note** There are two trivial cases:
1. If \( R(x) = 0 \) then the eqn is 1st order linear.

\[
\frac{dy}{dx} - Q(x)y = P(x)
\]

2. If \( P(x) = 0 \) then the eqn is Bernoulli

\[
\frac{dy}{dx} = Q(x)y + R(x)y^2
\]

Notice that our substitution \( y = u + y_p \) eliminated \( P(x) \). That was the whole point of the substitution, to eliminate the term that was a factor of \( x \) alone (the term not multiplied by \( y^n \), for \( n = 1, 2 \)).

Sometimes it can be advantageous to convert the Ricatti eqn to a 2nd order linear eqn. To do this let

\[
y(x) = -\frac{w'(x)}{R(x)w(x)}
\]

(Z)

Substituting (Z) into (1) yields

\[
w'' - \left( \frac{R'(x)}{R(x)} + Q(x) \right) w' + R(x)P(x)w = 0.
\]

In practice this is not very useful unless \( P, Q, \) and \( R \) are constants. The transformation can be used in reverse (this one can be more useful in practice from a point of view that it is easier to solve a Ricatti eqn numerically). To do this let

\[
\frac{w'}{w} = -R(x)y \Rightarrow \ln w = -\int_{x}^{\infty} R(x)y dx \Rightarrow w = e^{-\int_{x}^{\infty} R(x)y dx}
\]

\[
w(x) = e^{-\int \frac{R(x)}{R'} y(x) dx}
\]
Ricatti Eq's

\[ \text{Ex 1} \] Solve \( y' = y^2 - xy + 1 \).

\[ \text{Soln} \] Clearly this is a Riccati eq^n with \( P = 1, Q = -x, \) and \( R = 1. \)

Since the right hand side is a polynomial in \( x \) and \( y \) of degree 2, a natural guess might be \( y = ax^2 + bx + c. \) However, this will be a bit messy when sub's fitvted into the eq^n (We'll have an eq^n in powers of \( x \)). Let's not be so general to start with. Look for a soln of the form:

\[ y = ax^\alpha \]

\[ \text{Note:} \] The parameter \( a \) is necessary because the eq^n is nonlinear.

Then \( \frac{dy}{dx} = y^2 - xy + 1 \) becomes

\[ a\alpha x^{\alpha - 1} = a^{2\alpha} x^{2\alpha} - a^{\alpha + 1} + 1. \]

\[ \Rightarrow a (a^{\alpha - 1} - a^{2\alpha} x^{\alpha + 1}) = 1 \quad \text{(This must hold for every \( x \))} \quad (2) \]

\[ \text{It is not completely obvious what to do next. Let us try to get a pair of terms involving} \ x \ \text{to cancel.} \]

Want \( x^{\alpha + 1} - a^{2\alpha} = 0 \) \( \Leftrightarrow \alpha + 1 = 2\alpha \) and \( a = 1 \), \( \Leftrightarrow \alpha = 1 \) and \( a = 1 \).

Using \( \alpha = 1 \) and \( a = 1 \) in \( \text{eqn}(1) \) yields

\[ 1 \cdot (x^{1\cdot 1} - x^{2\cdot 1}) = x^0 + 0 = 1. \]

Thus, \( \text{\( y = x \)} \) is a particular soln.

To solve the eq^n we let

\[ \text{\( y = u + y_p = u + x \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 1 \)} \quad (3) \]

Substituting \( (3) \) into \( (1) \) yields

\[ \frac{du}{dx} + 1 = (u + x)^2 - x(u + x) + 1 = u^2 + 2ux + x^2 - ux - x^2 + 1 = u^2 + ux + 1 \]
Riccati Eq's

\[ \frac{du}{dx} = u^2 + ux \quad (\text{Bernoulli}) \]

\[ \frac{u^2}{u} \rightarrow u^{-2} \frac{du}{dx} - x u^{-1} = 1 \]  \hspace{1cm} (4)

Let \( w = -u^{-1} \), \( dw = -u^{-2} du \)

and substitute this into (4)

\[ \frac{dw}{dx} + x w = 1 \quad (\text{linear 1st-order}), \quad P(x) = x, \quad f(x) = -1 \]

Let \( u = e^\frac{\int x \, dx}{\frac{1}{2}} = e^{\frac{x^2}{2}} \).

\[ \rightarrow \frac{d}{dx} (e^{\frac{x^2}{2}} w) = e^{\frac{x^2}{2}} \]

\[ \int dx \quad e^{\frac{x^2}{2}} w = \int e^{\frac{x^2}{2}} \, dx + c \]

\[ \rightarrow w = e^{-\frac{x^2}{2}} \left[ \int e^{\frac{x^2}{2}} \, dx + c \right] \]

Back substitution \( w = -u^{-1} \)

\[ \frac{-1}{u} = \frac{\int e^{\frac{x^2}{2}} \, dx + c}{e^{\frac{x^2}{2}}} \]

\[ \rightarrow u = \frac{-e^{\frac{x^2}{2}}}{\int e^{\frac{x^2}{2}} \, dx + c} \]

Back substitution

\[ y = \frac{-e^{\frac{x^2}{2}}}{\int e^{\frac{x^2}{2}} \, dx + c} + x \]

Challenge problem: Solve \( y' = \frac{1}{x^4} - y^2 \).

The hard part will be finding a particular solution. The rest is straightforward.