Cauchy-Euler Eqn.s

We have just seen how to solve homogeneous linear constant coef. eqns. of the form
\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = 0, \]  
\[ (1) \]
where \( a_i(x) \) were all constants for \( i = 0, 1, \ldots, n \). \( (a_n(x) = a_i) \)

Next, we wish to exam a related problem. We want to examine ODEs of the form
\[ a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0, \]  
\[ (2) \]
where the coef. s are of the special form \( a_i(x) = a_i x^i \).
Notice that every term contains an \( x^i \frac{d^i}{dx^i} \) component.

**Def.** Any equation of the form
\[ a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = F(x), \]
where \( a_0, \ldots, a_n \) are constants and \( a_n \neq 0 \) is called a Cauchy-Euler eqn. of degree \( n \).

**Note:**
(1) Sometimes Cauchy-Euler eqn. s are just referred to as Euler eqn. s.
(2) In the case \( F(x) = 0 \) the eqn is referred to as an equidimensional, or scale invariant eqn. The term scale invariant comes from the fact that if you stretch or compress the independent variable by a scaling factor \( \lambda \) (i.e. let \( x \to \lambda x \), where \( \lambda \) is a constant), then the eqn remains invariant under this transformation.
We now demonstrate the scale-invariant property of the Cauchy-Euler equation.

Let \( z = \lambda x \), then \( y(x) = y(x(z)) \), where \( x = \frac{1}{\lambda} z \). Then
\[
\frac{d}{dz} y(x(z)) = \frac{dy}{dx} \frac{dx}{dz} = \frac{1}{\lambda} \frac{dy}{dx}.
\]
Thus \( \frac{d}{dz} = \frac{d}{d(\lambda x)} = \frac{1}{\lambda} \frac{d}{dx} \).

For all practical purposes we can factor the \( \lambda \) out of the denominator.
\[
\frac{d}{d(\lambda x)} = \frac{1}{\lambda} \frac{d}{dx} \implies \frac{d^m}{d(\lambda x)^m} = \frac{d}{d(\lambda x)} \cdot \frac{d}{d(\lambda x)} \cdots \frac{d}{d(\lambda x)} = \frac{1}{\lambda^m} \frac{d^m}{dx^m}.
\]

If we now let \( x \to \lambda x \) in the homogeneous Cauchy-Euler equations we get
\[
an (\lambda x)^n \frac{d^n y}{d(\lambda x)^n} + a_{n-1} (\lambda x)^{n-1} \frac{d^{n-1} y}{d(\lambda x)^{n-1}} + \ldots + a_1 (\lambda x) \frac{dy}{d(\lambda x)} + a_0 y = 0
\]
\[
\implies a_n \lambda^n x^n \frac{d^n y}{dx^n} + \ldots + a_1 \lambda \frac{dy}{dx} + a_0 y = 0
\]
all \( \lambda^i \)'s cancel
\[
a_n x^n \frac{d^n y}{dx^n} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = 0
\]
To find a method to solve for the general solution to the Cauchy-Euler Eqns., we will follow the strategy that we used to come up with the ansatz \( y = e^{mx} \) in the case of ODEs with constant coefficients.

We start with the case \( n = 1 \) and use separation of variables to find the general solution.

\[
a_1 x \frac{dy}{dx} + a_0 y = 0 \quad (a_i \neq 0) \quad (\text{Assume } x \neq 0 \text{ so } x > 0)
\]

\[
\Rightarrow \quad \frac{dy}{y} = -\frac{a_0}{a_1} \frac{dx}{x} \quad \Rightarrow \quad \ln y = -\frac{a_0}{a_1} \ln x + \ln c
\]

\[
\Rightarrow \quad y = c x^{-a_0/a_1}
\]

\( y = \) constant \( x^m \) is the general form of the solution.

Next, we make the ansatz \( y = x^m \) to look for a general solution to the \( n^{th} \)-order Cauchy-Euler eqns., where \( m \in \mathbb{R} \). Then

\[
y = x^m
\]

\[
y' = mx^{m-1}
\]

\[
y'' = m(m-1)x^{m-2}
\]

\[
\vdots
\]

\[
y^{(m)} = m(m-1)\cdots(m-n+1)x^{m-n}
\]

In the special case that \( m \in \mathbb{N} \), we would have

\[
y^{(m)} = m!, \quad y^{(m+1)} = y^{(m+2)} = \cdots = 0.
\]

**Fact:** It turns out that we can transform the Cauchy-Euler eqn.(1) into a constant coefficient eqn. using the change of variables:

\[
x = e^t. \quad \text{Under this transformation an } n^{th} \text{ degree Cauchy-Euler eqn. in } x \text{ will become a constant coefficient eqn. in } t.
\]

(See the next page for motivation.)
Cauchy-Euler eqn

In summary, for the case $n=2$, if

$$a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + cy = 0$$

then let

$$\begin{cases} y = x^m \\ y' = mx^{m-1} \\ y'' = m(m-1)x^{m-2} \end{cases} \quad \implies \quad \begin{cases} y = x^m \\ xy' = mx^m \\ x^2y'' = m(m-1)x^m \end{cases}$$

Substituting (2) into (1)

$$am(m-1)x^m + bmx^m + cx^m = 0$$

$$\therefore x^m \neq 0 \quad a(m^2-m) + bm + c = 0$$

$$a m^2 + (b-a)m + c = 0 \quad (The \ \text{Auxiliary \ eqn})$$

Notice that this is the same auxiliary eqn we would get if we started with the linear, homogeneous, constant-coefficient eqn

$$a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$$

To see this, just let $y = e^{mt}$. If we equate the two ansatzes $y = x^m$ and $y = e^{mt}$ then we get a relationship between $x$ and $t$:

$$\begin{cases} y = x^m = e^{\ln x^m} = e^{m \ln x} \\ y = e^{mt} \end{cases} \quad \implies \quad e^{m \ln x} = e^{mt}$$

$$\therefore \quad t = \frac{1}{m} \ln x$$

This is not a proof, just an observation. Let us now prove this.
Consider the 2nd order Cauchy–Euler eqn
\[ a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + c y = 0. \quad (1) \]

Let us transform this eqn to a constant coefficient eqn using the change of variables \( x = e^t \). Then \( x = x(t) \) and
\[
\frac{d}{dt} y(x(t)) = \frac{dy}{dx} \frac{dx}{dt} = e^t \frac{dy}{dx} = x \frac{dy}{dx}
\]

\[
\implies \frac{d}{dt} = x \frac{d}{dx} \quad (4)
\]

Next, we compute \( x^2 \frac{d^2}{dx^2} \) in terms of \( t \).
\[
\frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = x \frac{d}{dx} \left( x \frac{dy}{dx} \right) = x \frac{dy}{dx} + x^2 \frac{d^2 y}{dx^2}
\]
\[
= \frac{dy}{dt} + x^2 \frac{d^2 y}{dx^2}
\]

\[
\implies x^2 \frac{d}{dx} = \frac{d^2}{dt^2} - \frac{d}{dt} \quad (5)
\]

Notice that we have found the operator relationships between \( x \) and \( t \) indirectly. We start with the solution \( y = y(x) \) and we substitute \( x = x(t) \) into the soln to get a fun of \( t \): \( y = y(x(t)) \). Then taking the der. of this expression w.r.t. \( t \) (not \( x \)) we get a relationship between \( x \frac{d}{dx} \) and \( \frac{d}{dt} \). We never computed \( x \frac{d}{dx} \) directly.

The same indirect approach was used to get the higher der., as well.
Cauchy–Euler Eqns

Substituting (4) and (5) into (1) yields

\[ a \left( \frac{d^2}{dt^2} - \frac{d}{dt} \right) y + b \frac{dy}{dt} + cy = 0 \]

\[ \rightarrow a \frac{d^2y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0 \quad (6) \]

This is a constant-coefficient homogeneous linear ODE. Notice that eqn (6) is eqn (3). This verifies what we set out to show.

We now have a change of variables, \( x = e^t \) or \( t = \ln x \), that we can use to relate the fundamental 3 types of solutions for constant coef. linear ODEs with the solutions to the 2nd order Euler eqns. We summarize this connection between the fundamental solutions below. The connection is made by substituting \( t = \ln x \).

**Case 1** (Two distinct real roots \( m_1 < m_2 \))

\[ \{y_1, y_2\} = \{e^{m_1t}, e^{m_2t}\} = \{x^{m_1}, x^{m_2}\} \]

**Case 2** (Double root)

\[ \{y_1, y_2\} = \{e^{m_1t}, e^{m_1t}\} = \{x^{m_1}, (\ln x)x^{m_1}\} \]

**Case 3** (Complex conjugate roots: \( m_\pm = \alpha \pm i\beta \))

\[ \{y_1, y_2\} = \{e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)\} = \{x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)\} \]

Where the 1st set of eqns correspond to the fundamental set of the constant coef. linear ODE and the 2nd set of fundamental solutions belong to the corresponding Euler eqns.
Example 1 [distinct roots] Solve \( xy'' - y' = 0 \).

**Solution:**

Let \( y = e^{\ln x} = x^m \). Substituting this into the eqn yields

\[
\begin{align*}
  y' &= mx^{m-1} \\
  y'' &= m(m-1)x^{m-2}
\end{align*}
\]

\( x^m(m-1)x^{m-2} - mx^{m-1} = 0 \quad \Rightarrow \quad (m^2 - 2m)x^{m-1} = 0 \quad \Rightarrow \quad m(m-2) = 0 \)

\( \Rightarrow \) The fundamental set is \( \{y_1, y_2, y_3\} = \{x^0, x^2, x^{2/3}\} \).

\( \therefore \) \( y = c_1 + c_2x^2 \) is the general solution.

Example 2 [double root] \( x^2y'' + 5xy' + 4y = 0 \).

**Solution:**

Let \( y = x^m \)

Substituting this into the eqn yields

\( 0 = x^2y'' + 5xy' + 4y = m(m-1)x^m + 5mx^m + 4x^m = (m^2 + 4m + 4)x^m \)

\( \Rightarrow \) \( x^m \neq 0 \quad \Rightarrow \quad (m + 2)^2 = 0 \quad \Rightarrow \quad m = -2 \) (double root) \( \Rightarrow \) \( x^{-2} \) is a solution.

To get a 2nd independent solution, take \( \partial_m x^m \bigg|_{m = -2} = 1n x^{-2} \).

The fundamental set is \( \{y_1, y_2, y_3\} = \{x^{-2}, x^{-2}, 1n x^{-2}\} \).

\( \therefore \) \( y = c_1x^{-2} + c_2x^{-2} 1n x = \frac{c_1 + c_2 1n x}{x^2} \).

Example 3 [complex conjugate roots] Solve \( x^2y'' - 7xy' + 41y = 0 \).

Let \( y = x^m \), the auxiliary eqn becomes

\( 0 = m(m-1) - 7m + 41 = m^2 - 8m + 41 = (m - 4)^2 + 5^2 \)

\( \Rightarrow m = 4 \pm 5i \) (complex roots)

The fundamental set is \( \{y_1, y_2, y_3\} = \{e^{4 1n x} \cos(5 1n x), e^{4 1n x} \sin(5 1n x)\} \).

\( \therefore \) \( y = x^4(c_1 \cos(5 1n x) + c_2 \sin(5 1n x)) \).