Physics 210 Lab # 6:
Projectile Motion (The general case)

Wayne Hacker
Copyright ©Wayne Hacker 2011. All rights reserved.
1 Projectile motion

1.1 Getting Started

Before you read this lab

Watch the introductory video:

- How to use the spring cannon

Before you dare set foot in the lab

- Read this entire lab assignment, plus the instructions for all apparatus needed for the experiment.
- Prepare a spreadsheet based on the “Data analysis” section of these instructions. When you do the lab work, you should be able to enter your data into the spreadsheet and immediately see what values to use for later steps.

Apparatus

- Spring-cannon launcher and plastic ball
- Meter stick
- Sandbox
- Vertical ring and horizontal hoop with basket

Objectives

- To expose you to some of the complications that can arise when trying to measure and predict the behavior of a real physical system in a real lab.
- To give you experience writing spreadsheets for real-time use in a physics lab.
- To make measurements, and from them calculate the muzzle velocity of a ball shot from a spring cannon, using two different approaches.
- To determine the range of a projectile fired at some specific angle of elevation from some specific height above the ground.
- To determine the angle of elevation at which the spring cannon must be fired in order to hit a target at some specific distance and height.
- To predict the vertex of a ball fired from the spring cannon at a particular angle of elevation.
1.2 Introduction

This lab is concerned with projectile motion: the motion of an object launched at a certain speed and then subjected only to the constant acceleration of gravity. Real-world examples are everywhere: bullets shot from guns, erasers hurled across classrooms, basketballs thrown into hoops, bottles tossed into garbage cans, pianos flung by trebuchets...

Projectile motion is a good subject for study because it describes a lot of real-world situations, and because the math involved is not especially difficult. To keep the math from becoming difficult, we make a few reasonable assumptions.

The foremost of these assumptions is that gravity is the only force acting on the projectile. In particular, we assume that there is no air resistance. That’s a reasonable assumption, as long as the projectile is fairly dense and is not moving too fast through the air. It breaks down if the projectile isn’t dense enough (e.g. a loose wad of paper flung across a room) or if it’s moving far and fast enough to make air resistance a serious effect (e.g. an artillery shell fired a long distance at supersonic speed). These effects should not be a problem in this lab.

The lab will consist of three overall parts. In the first part, you will determine the speed of a ball as it emerges from a spring cannon, using two different experimental approaches. In the second, you will predict the range of a cannon fired at a specific angle of elevation from a specific height above ground level; you will also predict the vertex of the ball’s trajectory. In the third, you will calculate the angle of elevation necessary to hit a target at a specific height and distance; again, you will predict the vertex of the ball’s trajectory.

You should not assume that you will get the same value for the initial speed $v_0$ using these two methods. It is quite possible that you will get two significantly different results. If so, you should try to think of reasons why this should be the case, and what value or values you should use in Parts 2 and 3 of the lab.

You will find the theory used to calculate $v_0$ in section 1.3.1, beginning on page 7.
1.2.2 Part 2: Fixed angle of elevation

Once you know the initial speed $v_0$ of the projectile, you will fire it at an angle of elevation given you by your lab instructor. You will measure the height $h$ of the cannon above the ground; and you know the value of the gravitational acceleration $g$.

From these, you should be able to predict the location of the vertex, $(x_{\text{ymax}}, y_{\text{max}})$, which is the point at which the ball reaches its maximum height; and the range point, $(x_{\text{max}}, 0)$, which is the point where the ball strikes the ground. For the given value of $\theta_0$, you will calculate $x_{\text{ymax}}$, $y_{\text{max}}$, and $x_{\text{max}}$; you will then fire the cannon and see if the actual experiment justifies your predictions.

The theory that you will use to find the vertex is in section 1.3.2, beginning on page 10; the theory that you will use to find the range point is in section 1.3.2, beginning on page 11.

1.2.3 Part 3: Hitting the hoop

If you know the initial speed $v_0$ of the projectile, you can determine the angle of elevation at which you’ll need to fire it in order to hit a target at an arbitrary horizontal distance and height above the ground, or whether the target is out of range of the projectile.

The lab attendant will provide you with such a target, in the form of a hoop on a stand. You will calculate the angle of elevation for this target. The theory that you will use is in section 1.3.2, beginning on page 13.

Given the angle of elevation, you can predict the location of the vertex, $(x_{\text{ymax}}, y_{\text{max}})$, which is the point at which the ball reaches its maximum height. You will perform this calculation and then place a vertical hoop at your predicted vertex.

You will then test your predictions by firing the cannon at your predicted angle of elevation. If your calculations are correct, the ball should go through the vertical hoop at the vertex, then land in the basket suspended below the target hoop.

1.2.4 Part 0: Spreadsheet

Before you arrive at the lab, you should prepare a spreadsheet based on the instructions in section 1.5, beginning on page 20.

You should make sure that your spreadsheet is working correctly before you come into the lab. You will need to plug the data from Part 1 into the spreadsheet in order to find the values that you’ll need for Part 2 and Part 3. Do not collect data for Part 1, leave the lab, and then return to do Parts 2 and 3. You should do this entire experiment in a single lab session. If your spreadsheet is put together correctly, you
should have no trouble doing this. If it’s not, you should fix the spreadsheet before you begin Part 1.

To make sure that your spreadsheet is working properly, you should test all parts of it very thoroughly before you come to the lab. If you’re working in a group, then everyone in the group should independently come up with a set of test numbers. Test every part of the spreadsheet separately: the two calculations of \( v_0 \) from Part 1; the calculations of range and vertex from Part 2; and the calculation of \( \theta_0 \) and of the vertex from Part 3. Thorough testing is essential: you don’t want to discover problems with your spreadsheet while you’re in the middle of the experiment.

### 1.3 Theory

We are looking at the general case in which a projectile is launched, not necessarily from ground level, at some angle of elevation. As the projectile moves horizontally, it rises to a maximum height, then falls again until it strikes the ground. As a concrete example, consider a spring cannon sitting on a tabletop above ground level, and fired upward at an angle \( \theta_0 \) above the horizontal.

We will use the notation shown in Figure 1 and the table following it.

![Figure 1: Trajectory of a ball fired at time \( t = 0 \), with initial speed \( v_0 \), from a spring cannon located at \( (0, h) \) with an elevation angle of \( \theta_0 \).](image-url)
The projectile is launched at time $t = 0$ from initial position $(x_0, y_0) = (0, h)$, at an angle of elevation $\theta_0$ and with initial speed $v_0$. The projectile will gain height (increasing $y$) until time $t_{y_{\text{max}}}$. At time $t = t_{y_{\text{max}}}$, it will reach its maximum height of $y = y_{\text{max}}$; at this point, its horizontal position will be $x = x_{y_{\text{max}}}$. It will then lose height (decreasing $y$) until it strikes the ground at time $t = t_{x_{\text{max}}}$. At this point, its position will be $(x_{\text{max}}, 0)$. We will often refer to $x_{\text{max}}$ as the range of the projectile.

We will assume that $0 \leq \theta_0 \leq 90^\circ$. We will assume that $h \geq 0$; however, the analysis would be very similar if $h$ were negative (if, for instance, we were firing a spring cannon from the floor and having the ball land on a tabletop whose height was taken as $y = 0$).

The fundamental kinetic equations govern the motion of the projectile:

$$v = v_0 + at \quad (1)$$
$$v^2 = v_0^2 + 2a(x - x_0) \quad (2)$$
$$x = x_0 + v_0t + \frac{1}{2}at^2 \quad (3)$$

As it turns out, we won’t need equation (2) to analyze projectile motion. We will use equations (1) and (3) twice: one set of equations for the horizontal component of the motion (the $x$-direction), and one set for the vertical component (the $y$-direction). For the $x$-direction,

$$v_x = v_{0x} + a_xt \quad (4)$$
$$x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2 \quad (5)$$

For the $y$-direction,

$$v_y = v_{0y} + a_yt \quad (6)$$
$$y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2 \quad (7)$$

We’ve used $x$ and $y$ subscripts to distinguish the horizontal and vertical components of $v$, $v_0$, and $a$.

We assume that there is no air resistance, and that after the projectile is launched the only force acting on it is gravity. Since there’s no force in the $x$-direction, $a_x = 0$. Since upward is the positive direction, $a_y = -g$. We are launching the ball from initial position $(x_0, y_0) = (0, h)$. 

<table>
<thead>
<tr>
<th>Event</th>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial launch</td>
<td>0</td>
<td>0</td>
<td>$h$</td>
</tr>
<tr>
<td>Maximum height (vertex)</td>
<td>$t_{y_{\text{max}}}$</td>
<td>$x_{y_{\text{max}}}$</td>
<td>$y_{\text{max}}$</td>
</tr>
<tr>
<td>Hits ground (range point)</td>
<td>$t_{x_{\text{max}}}$</td>
<td>$x_{\text{max}}$</td>
<td>0</td>
</tr>
</tbody>
</table>
In our $x$-direction equations (4) and (5), we substitute $a_x = 0$ and $x_0 = 0$ to get

$$v_x = v_{0x}$$
$$x = v_{0x}t$$ \hspace{1cm} (8)

In our $y$-direction equations (6) and (7), we substitute $a_y = -g$ and $y_0 = h$ to get

$$v_y = v_{0y} - gt$$
$$y = h + v_{0y}t - \frac{1}{2}gt^2$$ \hspace{1cm} (10)

Since we know the launch speed $v_0$ and the angle of elevation $\theta_0$, we can calculate $v_{0x}$ and $v_{0y}$. See the figure at right:

$$v_{0x} = v_0 \cos \theta_0$$ \hspace{1cm} (12)
$$v_{0y} = v_0 \sin \theta_0$$ \hspace{1cm} (13)

We won’t substitute these into our equations of motion yet, because at times it will be easier to explain things in terms of $v_{0x}$ and $v_{0y}$.

We are interested in two particular points on the trajectory. The first is the vertex, located at $(x_{\text{ymax}}, y_{\text{max}})$. The second is the range point, $(x_{\text{max}}, 0)$, at which the projectile strikes the ground. Lastly, we’ll find an equation for the trajectory of the ball in the $xy$-plane: $y = y(x)$. This will allow us to determine what value of $\theta_0$ to use if we want to shoot the ball through a hoop located at position $(x_{\text{hoop}}, y_{\text{hoop}})$.

### 1.3.1 Special cases: horizontal and vertical firing

We’ll begin by looking at two special cases, which you’ll use to determine $v_0$ in this lab: the case in which the projectile is fired horizontally ($\theta_0 = 0$); and the case in which the projectile is fired straight upward ($\theta_0 = 90^\circ = \pi/2$ rad).

**Special case 1: horizontal firing**

First, let’s look at the case in which we fire the projectile horizontally, i.e. with $\theta_0 = 0$, from an initial position of $(x_0, y_0) = (0, h)$, where $h > 0$. The situation is sketched in Figure 2, below.

We’ll again use the fundamental kinematic equations to analyze the motion of the ball. In the $x$-direction, $x_0 = 0$ and $a_x = 0$. By equation (12), $v_{0x} = v_0 \cos(0) = v_0$. Substituting these into equation (5) gives us

$$x = v_0t$$ \hspace{1cm} (14)
In the $y$-direction, $y_0 = h$ and $a_y = -g$. By equation (13), $v_{0y} = v_0 \sin(0) = 0$. Substituting these into equation (7) gives us

$$y = h - \frac{1}{2}gt^2$$

We can use equation (15) to find $t_{\text{max}}$:

$$y(t_{\text{max}}) = 0 = h - \frac{1}{2}gt_{\text{max}}^2 \Rightarrow t_{\text{max}} = \sqrt{\frac{2h}{g}}$$

Substituting this into equation (14) gives

$$x_{\text{max}} = x(t_{\text{max}}) = v_0 \sqrt{\frac{2h}{g}}$$

If we can measure $x_{\text{max}}$ and $h$, and if we know $g$, then we can use this equation to find $v_0$:

$$v_0 = \sqrt{\frac{g}{2h} x_{\text{max}}}$$

Special case 2: vertical firing

Next, we’ll look at the special case in which the projectile is fired straight upward. In this case, $\theta_0 = 90^\circ = \pi/2$ radians. Then by equations (12) and (13), $v_{0x} = 0$ and $v_{0y} = v_0$. 
We don’t need to consider the equations of motion in the \( x \)-direction at all: since \( v_{0x} = 0 \) and \( a_x = 0 \), the projectile doesn’t move at all in the \( x \)-direction, so its \( x \)-coordinate is always zero.

It doesn’t make sense to look for the range point, since \( x = 0 \) for all time. However, the highest point, or vertex \((0, y_{\text{max}})\), is of interest.

In the \( y \)-direction, equations (10) and (11) become

\[
\begin{align*}
v_y &= v_0 - gt \\
y &= h + v_0 t - \frac{1}{2} gt^2
\end{align*}
\]

(17)

(18)

At the projectile’s highest point, its velocity \( v_y \) in the \( y \)-direction is zero. This is because at the highest point, \( dy/dt = v_y = 0 \). We’ll use this to find the time \( t_{y_{\text{max}}} \) at which the highest point is reached; then we’ll substitute that time into equation (18) to get \( y_{\text{max}} \).

At \( t = t_{y_{\text{max}}} \), we know that \( v_y = 0 \). Substituting these into equation (17) gives us

\[
0 = v_0 - gt_{y_{\text{max}}} \quad \Rightarrow \quad t_{y_{\text{max}}} = \frac{v_0}{g}
\]

Substitute this into equation (18):

\[
y_{\text{max}} = y(t_{y_{\text{max}}}) = h + \frac{v_0^2}{2g}
\]

(19)

In Part 1 of the lab, you will use this as a second means of determining \( v_0 \). You will fire the cannon from a known height \( h \) and measure the maximum height \( y_{\text{max}} \). If we know \( h \) and \( y_{\text{max}} \), we can use equation (19) to find \( v_0 \):

\[
v_0 = \sqrt{2g(y_{\text{max}} - h)}.
\]

(20)

We will typically put one end of the ruler on the mouth of the cannon; this will allow us to use \( h = 0 \) in equation (20).

### 1.3.2 The general case

Now, we’ll look at the general case, where \( 0 \leq \theta_0 \leq 90^\circ \). The techniques that we use to find the vertex and the range point will be very similar to the ones we used to find the vertex for the case of vertical firing, and the range point for the case of horizontal firing.
The vertex

We’ll start by finding the vertex. The critical thing to note is that at the highest point of the trajectory, the y-component of the projectile’s velocity must be zero: \( v_y = 0 \). Again, this is because at the highest point, \( dy/dt = v_y = 0 \).

We will use \( t_{ymax} \) to denote the time at which the projectile reaches the vertex \((x_{ymax}, y_{max})\). Then \( v_y(t_{ymax}) = 0 \). Substituting these into equation (10) leads us to a formula for \( t_{ymax} \):

\[
0 = v_{oy} - gt_{ymax} \quad \Rightarrow \quad t_{ymax} = \frac{v_{oy}}{g}
\]

This is our first useful result: how much time it takes for the projectile to travel from its initial launch to the high point of its trajectory. We can now use \( t = t_{ymax} \) in equations (9) and (11) to calculate \( x_{ymax} \) and \( y_{max} \) respectively:

\[
x_{ymax} = x(t = t_{ymax}) = v_{ox} t_{ymax} = \frac{v_{ox} v_{oy}}{g}
\]

\[
y_{max} = y(t = t_{ymax}) = h + v_{oy} t_{ymax} - \frac{1}{2}gt_{ymax}^2 = h + \frac{v_{oy}^2}{2g}
\]

We summarize our results for the vertex, given the initial velocity in component form:

\[
t_{ymax} = \frac{v_{oy}}{g} \quad \quad (21)
\]

\[
x_{ymax} = \frac{v_{ox} v_{oy}}{g} \quad \quad (22)
\]

\[
y_{max} = h + \frac{v_{oy}^2}{2g} \quad \quad (23)
\]

In the lab, we usually don’t measure \( v_{ox} \) and \( v_{oy} \) directly; we measure \( v_0 \) and \( \theta_0 \), and have to calculate \( v_{ox} \) and \( v_{oy} \) using equations (12) and (13). If we substitute these into equations (21), (22), and (23), we get

\[
t_{ymax} = \frac{v_0 \sin \theta_0}{g} \quad \quad (24)
\]

\[
x_{ymax} = \frac{v_0^2 \cos \theta_0 \sin \theta_0}{g} = \frac{v_0^2}{2g} \sin(2\theta_0) \quad \quad (25)
\]

\[
y_{max} = h + \frac{v_0^2}{2g} \sin^2 \theta_0 \quad \quad (26)
\]

In equation (25), we used the trig identity \( 2 \sin \theta_0 \cos \theta_0 = \sin(2\theta_0) \) to simplify the expression for \( x_{ymax} \).
The range point

The other point we’re interested in is the range point: where the projectile strikes the ground at \((x_{\text{max}}, 0)\), at time \(t_{\text{max}}\). We will use an approach like the one we used to find the vertex: we will begin by finding \(t_{\text{max}}\), then find \(x_{\text{max}} = x(t = t_{\text{max}})\).

We know that at the range point, \(y = 0\). We will use that and equation (11) to find the value of \(t_{\text{max}}\):

\[
y(t = t_{\text{max}}) = 0 = h + v_0 t_{\text{max}} - \frac{1}{2} g t_{\text{max}}^2
\]  

(27)

We can solve this for \(t_{\text{max}}\) using the quadratic formula:

\[
t_{\text{max}} = \frac{v_0 + \sqrt{v_0^2 + 2gh}}{g}
\]  

(28)

A quadratic equation can have as many as two distinct real solutions, and ours has two. Since \(v_0, g,\) and \(h\) are all assumed positive, the quantity \(v_0^2 + 2gh\) under the square root in equation (28) is positive. (If it had been negative, there wouldn’t have been any real solutions; if it had been zero, there would have been only one real solution.)

This looks like a problem. If you look at Figure 1, the ball only crosses the \(x\)-axis once. But we found two distinct solutions, given by equation (28), which indicates that the ball crosses the \(x\)-axis at two different times.

Happily, this isn’t a problem at all. Look more closely at the numerator in equation (28). In particular, look at the expression under the square root, bearing in mind that \(v_0, g,\) and \(h\) are all positive.

\[
\sqrt{v_0^2 + 2gh} < v_0\sqrt{v_0^2 + 2gh} \quad \text{(see “Technical point”, below)}
\]

\[
\Rightarrow \quad v_0 - \sqrt{v_0^2 + 2gh} < 0
\]

\[
\Rightarrow \quad \frac{v_0 - \sqrt{v_0^2 + 2gh}}{g} < 0
\]  

(29)

This means that of the two solutions for \(t_{\text{max}}\) given by equation (28), one is negative. We get it because the equations of motion don’t “know” that the ball started moving at \(t = 0\); as far as they’re concerned, the ball always has been, and always will be, moving on the same course; at time \(t = 0\), it passed through the point \((x, y) = (0, h)\), which happened to be where a spring cannon was located. If you extend the trajectory curve in Figure 1 to the left (later in this discussion, we will analyze it and see that it’s a parabola), you’ll see that it passes through the \(x\)-axis.
**Technical point:** In the second step above, we took the square root across an inequality. In general, for an arbitrary function \( f(x) \), we cannot assume that if \( a < b \), then \( f(a) < f(b) \). Functions for which this holds true, such as \( f(x) = \sqrt{x} \), are called *strictly monotonic increasing* functions.

Since we’re only interested in what happens after the cannon is fired at \( t = 0 \), we can ignore the negative solution. Thus we get a single value for \( t_{\text{max}} \):

\[
t_{\text{max}} = \frac{v_0 y + \sqrt{v_0^2 y^2 + 2gh}}{g} = \frac{1}{g} \left[ v_0 y + \sqrt{v_0^2 y^2 + 2gh} \right]
\]

(30)

Now we can find \( x_{\text{max}} \) by substituting \( t = t_{\text{max}} \) in equation (9).

\[
x_{\text{max}} = x(t = t_{\text{max}}) = v_0 x t_{\text{max}} = \frac{v_0 x}{g} \left[ v_0 y + \sqrt{v_0^2 y^2 + 2gh} \right]
\]

(31)

Again, we probably won’t have direct measurements of \( v_{0x} \) and \( v_{0y} \); we’ll have to calculate them from \( v_0 \) and \( \theta_0 \), using equations (12) and (13). Substituting these into equations (30) and (31) gives

\[
t_{\text{max}} = \frac{1}{g} \left[ v_0 \sin \theta_0 + \sqrt{v_0^2 \sin^2 \theta_0 + 2gh} \right]
\]

(32)

\[
x_{\text{max}} = \frac{v_0 \cos \theta_0}{g} \left[ v_0 \sin \theta_0 + \sqrt{v_0^2 \sin^2 \theta_0 + 2gh} \right]
\]

(33)

**Determining the trajectory:** \( y = y(x) \)

Our last objective is to find an equation for the trajectory of the ball in \( xy \)-space; that is, to find a curve of \( y \) as a function of \( x \), like the one shown in Figure 1.

We have equations for \( x \) and \( y \) as functions of \( t \). Equation (9) gives us \( x(t) \); equation (11) gives us \( y(t) \). To find \( y(x) \), we’ll solve equation (9) for \( t \), then substitute that expression into equation (11).

Begin by solving equation (9) for \( t \):

\[
x = v_0 x t \quad \overset{t = \frac{x}{v_0 x}}{\longrightarrow} \quad t = \frac{x}{v_0 x}
\]
Substitute into equation (11):

\[ y = h + v_0 y \frac{x}{v_0 x} - \frac{1}{2} g \left( \frac{x}{v_0 x} \right)^2 \]

\[ = h + v_0 y \frac{x}{v_0 x} - \frac{g}{2v_0^2} x^2 \]

\[ = h + v_0 \sin \theta_0 \frac{x}{v_0 \cos \theta_0} - \frac{g}{2v_0^2} \cos^2 \theta_0 x^2 \]

\[ = h + \tan \theta_0 x - \frac{g}{2v_0^2} \sec^2 \theta_0 x^2 \]

\[ y(x) = h + \tan \theta_0 x - \frac{g}{2v_0^2} (1 + \tan^2 \theta_0) x^2 \] \hspace{1cm} (34)

In the last two steps, we used two trig identities:

\[ \frac{\sin \theta}{\cos \theta} = \tan \theta \quad \text{and} \quad \sec^2 \theta = 1 + \tan^2 \theta \]

Given the location of the target, find the launch angle \( \theta_0 \)

Suppose now that we have a cannon with a fixed muzzle velocity \( v_0 \) and an adjustable angle of elevation \( \theta_0 \); and that we want to fire a ball and have it pass through a hoop whose center is at \((x_{\text{hoop}}, y_{\text{hoop}})\). What value of \( \theta_0 \) should we use?

We can find \( \theta_0 \) by setting \( x = x_{\text{hoop}} \) and \( y_{\text{hoop}} = y(x_{\text{hoop}}) \) in equation (34). That will give us a quadratic equation in \( \tan \theta_0 \). We can solve that quadratic, then take the inverse tangent to find \( \theta_0 \).

\[ y(x_{\text{hoop}}) = H_{\text{cannon}} + \tan \theta_0 x_{\text{hoop}} - \frac{g}{2v_0^2} (1 + \tan^2 \theta_0) x_{\text{hoop}}^2 \]

\[ \Rightarrow -y_{\text{hoop}} \rightarrow - \frac{g}{2v_0^2} (1 + \tan^2 \theta_0) x_{\text{hoop}}^2 + \tan \theta_0 x_{\text{hoop}} + H_{\text{cannon}} - y_{\text{hoop}} = 0 \]

Re-arranging terms, we see that we have a quadratic equation in \( \tan \theta_0 \)

\[ - \frac{g x_{\text{hoop}}^2}{2v_0^2} \left( \tan \theta_0 \right)^2 + x_{\text{hoop}} \tan \theta_0 + \left( H_{\text{cannon}} - y_{\text{hoop}} - \frac{g x_{\text{hoop}}^2}{2v_0^2} \right) = 0 \] \hspace{1cm} (35)

This equation may look daunting; but we’ll approach it as we would any other quadratic equation. Begin by making the obvious substitutions:

\[ X = \tan \theta_0 \quad \text{and} \quad \begin{cases} a = - \frac{g x_{\text{hoop}}^2}{2v_0^2} < 0 \quad \text{(the parabola points downward)} \\ b = x_{\text{hoop}} > 0 \\ c = \left( H_{\text{cannon}} - y_{\text{hoop}} - \frac{g x_{\text{hoop}}^2}{2v_0^2} \right) = (\Delta h - a) \end{cases} \] \hspace{1cm} (36)
where $\Delta h \equiv H_{\text{cannon}} - y_{\text{hoop}}$, the height of the cannon above the level of the hoop.

Substituting (36) into equation (35) yields a familiar-looking equation:

$$aX^2 + bX + c = 0,$$

which can easily be solved using the quadratic formula:

$$X_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Back-substituting $X_{\pm} = \tan(\theta_0^{(\pm)})$ and taking the inverse tangent of both sides yields the desired solution.

$$\tan(\theta_0^{(\pm)}) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \theta_0^{(\pm)} = \arctan \left( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

(37)

The quadratic equation can have zero, one, or two real solutions. In fact, there will often be two angles $\theta_0$ that will both allow the projectile to hit the target. For practical reasons, we will want to use the largest angle. One of these reasons is very practical: higher launch angles mean a steeper trajectory, which means that if the ball misses the basket, it will bounce at a higher angle, which means less chasing the ball around the lab! The other is a bit more subtle.

Our model assumes that the hoop is infinitely thin, and that the ball is a point particle. However, this is not the case in the real world. The ball is more likely to pass through the hoop without hitting the rim if its trajectory intersects the plane of the hoop at a high angle. For this reason, we will want to use the largest possible solution angle. But which one is it? To answer this question we’ll need two facts: (i) the arctangent, or inverse tangent, function is a monotonic increasing function (i.e., if $x_1 < x_2$, then $\tan^{-1}(x_1) < \tan^{-1}(x_2)$), and (ii) the parameter $a$ is negative (the parabola opens downward). Thus the bigger argument of arctangent, i.e. the bigger root of the quadratic, will yield the desired bigger launch angle.

Factoring a minus sign out of the numerator of the argument in equation (37) yields

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1}{2a} \left[ b \mp \sqrt{b^2 - 4ac} \right].$$

(38)

Since $-1/(2a) > 0$, it follows that the + case yields the desired root. Thus, the desired launch angle is

$$\theta_0^{(-)} = \arctan \left( \frac{1}{2a} \left[ b + \sqrt{b^2 - 4ac} \right] \right).$$

(39)

In order to be able to use this formula in the lab in real time, you will need to “code up” this expression in your spreadsheet. We strongly recommend that you code the formula up the way it is written by making a cell for each of the parameters $a$, $b$, and $c$, and also
that you name these cells A, B, and C. (Unfortunately, neither Excel nor Calc will allow you to name a cell “C”, so you have to use some substitute.) Then, when you code the formula in the spreadsheet, it will look like the above formula. This reduces the chance of error. Do not try to back-substitute all of the original parameters into the solution formula (39). This would be a recipe for a debugging nightmare!

**Determining the maximum range for the hoop \(x_{\text{hoop,max}}\)**

It turns out that we can use equation (39) to check our code. Notice that the equation fails to give real-valued answers when the discriminant \(b^2 - 4ac < 0\). Thus, in order to have real-valued solutions, we need a non-negative discriminant: \(b^2 - 4ac \geq 0\). Substituting the values for the parameters \(a\), \(b\), and \(c\) in (36) we can find an upper bound on \(x_{\text{hoop}}\). We will denote this upper bound by \(x_{\text{hoop,max}}\).

Substituting for \(a\), \(b\), and \(c\) in \(b^2 - 4ac \geq 0\) yields

\[
x_{\text{hoop}}^2 - 4 \left( - \frac{gx_{\text{hoop}}^2}{2v_0^2} \right) \left( \Delta h - \frac{gx_{\text{hoop}}^2}{2v_0^2} \right) \geq 0
\]

\[
\rightarrow 1 + 2 \frac{g}{v_0^2} \left( \Delta h - \frac{gx_{\text{hoop}}^2}{2v_0^2} \right) \geq 0
\]

\[
\rightarrow 1 + 2g \Delta h - \frac{gx_{\text{hoop}}^2}{v_0^2} \geq 0
\]

\[
\rightarrow 1 + \frac{g \Delta h}{v_0^2} \geq \frac{g^2 x_{\text{hoop}}^2}{v_0^2}
\]

\[
\rightarrow \frac{v_0^4}{g} \left( 1 + \frac{g \Delta h}{1/2v_0^2} \right) \geq x_{\text{hoop}}^2
\]

\[
\rightarrow \frac{v_0^2}{g} \sqrt{1 + \frac{g \Delta h}{1/2v_0^2}} \geq x_{\text{hoop}}
\]

Notice that in the last step we applied the square root function across the inequality. This step is justified by the fact that \(f(x) = \sqrt{x}\) is a monotonic increasing function.

Setting \(x_{\text{hoop}}\) equal to the upper bound gives \(x_{\text{hoop,max}}\):

\[
x_{\text{hoop,max}} = \frac{v_0^2}{g} \sqrt{1 + \frac{g \Delta h}{1/2v_0^2}}.
\]

You should code up the value of \(x_{\text{hoop,max}}\) and set \(x_{\text{hoop}} > x_{\text{hoop,max}}\) in your code, so that the code will give you an error message if you try to hit a hoop that’s out of range of the cannon. Basic rule of thumb when writing code: Always use benchmarks, and the more the better!

Also, notice that the two solutions will be equal at the maximum range of the spring cannon (i.e., at the angle where \(\theta_0^{(-)} = \theta_0^{(+)}\)). Setting these two solutions equal requires
that the discriminant vanish. This condition leads to the useful result

\[ x_{\text{hoop, max}} = \sqrt{4ac} = 2\sqrt{ac}. \]  \hspace{1cm} (42)

You should use this result to check your code. If you choose \( x_{\text{hoop, max}} = \sqrt{4ac} = 2\sqrt{ac} \) and \( \Delta h = 0 \), so that you have the case of the full parabola, then your code should give \( \theta_0 = 45^\circ \) for the answer. This is a second check on your results!

**Comment:**

- Look at the quantity under the square root of equation (41). If we multiply both numerator and denominator by the mass of the ball \( m \), which will not change the value of the ratio, then we get the ratio of the initial potential energy to the initial kinetic energy of the ball:

\[ \frac{mg\Delta h}{\frac{1}{2}mv_0^2} = \frac{\text{P.E.}}{\text{K.E.}} . \]

This result shows that there is a direct relationship between the speed of the ball, the strength of the gravitational attraction, and the maximum range of the projectile. Indeed, we can see this in the equation by considering several extreme cases.

First, fix \( v_0 \) and consider what happens when \( g \) goes to zero:

\[ x_{\text{hoop, max}} = \lim_{g \to 0} \frac{v_0^2}{g} \sqrt{1 + \frac{g\Delta h}{\frac{1}{2}v_0^2}} = \lim_{g \to 0} \frac{v_0^2}{g} = \infty . \]

This is exactly the result we'd except from Newton’s first law: if we fired a spring cannon in interstellar space then it would travel forever.

Now, fix \( v_0 \) and consider what happens when \( g \) goes to infinity:

\[ x_{\text{hoop, max}} = \lim_{g \to \infty} \frac{v_0^2}{g} \sqrt{1 + \frac{g\Delta h}{\frac{1}{2}v_0^2}} = \lim_{g \to \infty} \frac{\text{constant}}{\sqrt{g}} = 0 . \]

Again, this is exactly what we’d expect: if we fired the spring cannon on the surface of a neutron star, the ball would travel almost no distance at all before hitting the ground.

Of course, we could also consider fixing \( g \) and changing the speed of the ball as it leaves the mouth of the spring cannon. This could actually be checked in the lab by using one, two, and three click-settings on the spring cannon. You should verify that the following limits are correct, and that they make physical sense.

\[ x_{\text{hoop, max}} = \lim_{v_0 \to \infty} \frac{v_0^2}{g} \sqrt{1 + \frac{g\Delta h}{\frac{1}{2}v_0^2}} = \infty \quad (g \text{ and } \Delta h \text{ are fixed}) \]

\[ x_{\text{hoop, max}} = \lim_{v_0 \to 0} \frac{v_0^2}{g} \sqrt{1 + \frac{g\Delta h}{\frac{1}{2}v_0^2}} = 0 \quad (g \text{ and } \Delta h \text{ are fixed}) \]
1.4 Lab procedure

In writing this section, we assume that you have watched the video that shows how to operate the spring cannon.

**Use the low setting.** The spring cannon has three settings: one click for low, two clicks for medium, three clicks for high. Use the low setting. Also, use the low setting. Moreover, *use the low setting*. If you’re not sure about this, ask the lab attendant, who will tell you to *use the low setting*.

**Measure the ball.** In the “Theory” section, we treated the ball like a point particle. In the real world, the ball has a positive diameter $d_{\text{ball}}$, and consequently a positive radius $r_{\text{ball}} = d_{\text{ball}}/2$. When the ball strikes the ground after being fired, its center is at a height of $r_{\text{ball}}$ above ground level. You need to take this into account when you are doing your calculations. For that reason, at the beginning of the experiment you should measure the ball. The easiest way to do this is to measure $d_{\text{ball}}$, then divide by two to get $r_{\text{ball}}$.

**Measure the height of the cannon.** You should measure the vertical distance $h_{\text{cm}}$ from the tabletop to the center mark on the cannon. This will be the height of the center of the ball when it emerges from the cannon. Because of the radius of the ball is $r_{\text{ball}} > 0$, you’ll have to modify $h_{\text{cm}}$ in various ways to get the value of $h$ that you’ll use in the formulae. We will discuss these modifications in the various steps of the experiment.

Since the cannon swivels about the center mark, you shouldn’t need to re-measure the height of the mark when you change the angle of the cannon.

1.4.1 Part 1: Initial velocity

In Part 1, you will determine the ball’s initial speed $v_0$ in two different ways: by firing the cannon horizontally and measuring the range $x_{\text{max}}$; and by firing it vertically and measuring the maximum height $y_{\text{max}}$.

You should not be surprised, or take it as a sign that you’ve done something wrong, if you get different results from these two approaches.

**Horizontal firing**

Make sure that the cannon is exactly horizontal. Shoot the ball and observe about where it lands on the table. You will probably need at least two people for this: one to operate the cannon, and one to note where the ball lands (and to catch the ball so that it won’t bounce all over the lab, disturbing other students as you chase it).

Now, place the sandbox so that the ball should land reasonably close to its center. Make sure that the sand is level, and measure the vertical distance $y_s$ from the tabletop to the
top of the sand layer. Now measure the vertical distance \( y_c \) from the tabletop to the center mark on the cannon.

Shoot the ball horizontally from the cannon; then measure the horizontal distance from the center mark on the cannon to the center of the crater in the sand where the ball landed. This horizontal distance is the range \( x_{\text{max}} \).

Repeat this a total of three times. It’s best if a different person measures \( x_{\text{max}} \) on each of the three trials. That way, if one person’s making a systematic error, it won’t be repeated. Use the average of the three values of \( x_{\text{max}} \) in your calculations.

When the ball was in the cannon, its center was at \( y_c \), the level of the center mark on the cannon. When it hits the sand, its center is \( r_b \) above the top of the sand, so \( y_s + r_b \) above the tabletop. Thus the total distance that the center of the ball has fallen is

\[
h = y_c - (y_s + r_b) = y_c - y_s - r_b
\]

In the data-analysis stage of this lab, you will use the average of the three \( x_{\text{max}} \) values, the vertical distance \( h \) from equation (43), and \( g = 9.79 \text{ m/s}^2 \) to calculate \( v_0 \) from equation (16).

**Vertical firing**

Now, turn the cannon so that it points straight upward. The cannon is designed so that when the ball leaves the spring, its leading edge is even with the mouth of the cannon. This is fortunate, because it means you don’t have to account for the radius of the ball. The height of the top of the ball above the cannon’s mouth is the same as the height of the center of the ball above \( h_{\text{cm}} \).

Hold a meter-stick vertically, with its zero mark at the cannon’s mouth. Make sure that it is as exactly vertical as you can get it; if it’s tilted, your measurement will be off.

Now, fire the cannon and see about how high the ball goes. Place your hand at or a little above that level, and fire the cannon again. (You will definitely need more than one person for this.) Repeat the process, moving your hand up and down until the top of the ball barely touches it at the top of its flight. Use the height of your hand, as measured by the meter-stick, as \( y_{\text{max}} \).

Repeat this a total of three times. Again, it’s best to have a different person measure \( y_{\text{max}} \) in each of the three trials. The person who’s making the measurement should not know the values of earlier measurements; that way, each measurement will be truly independent. Use the average of the three measured values of \( y_{\text{max}} \) in your calculations.

Since the zero mark on the meter stick is at the level of the top of the ball when it emerges from the cannon, and the height you’re measuring is that of the top of the ball at its highest point, you’ll use \( h = 0 \) in equation (20); so

\[
v_0 = \sqrt{2gy_{\text{max}}}
\]
1.4.2 Part 2: Fixed angle of elevation

Now, you’ll try to find the range and vertex for a fixed angle of elevation $\theta_0$.

You will be given a specific angle $\theta_0$; and a target, in the form of a horizontal hoop at a certain height with a basket suspended under it, will be given to you. The height of the target hoop should be lower than the height of the cannon. You will need to calculate the horizontal distance at which to place the hoop so that a ball fired from the cannon at the specified elevation $\theta_0$ will pass through the hoop and land in the basket. You should not adjust the height of the hoop; its height is part of the problem you are given.

You can treat this as a problem of finding the range $x_{\text{max}}$ of the ball, using an appropriate value of $h$. You will need a value of $v_0$ from the first part of the experiment. Since it’s likely that you will come up with different values of $v_0$ for the horizontal and vertical firings in the first part, you should think about how you might come up with a value of $v_0$ for an arbitrary angle of elevation $\theta_0$.

You will also need to calculate the vertex of the trajectory for the given $\theta_0$. You will place a vertical hoop with its center at your calculated vertex.

The test of your calculations will be firing the ball at the specified angle. If your calculations are correct, the ball should pass through the vertical hoop at the vertex, then through the horizontal hoop and into the basket below it.

The lab attendant will need to verify that you have successfully shot the ball through the vertex hoop and the horizontal hoop. He may also inspect your spreadsheet. In particular, he may give you values of $v_0$, $\theta_0$, and $h$, and require that you show that your spreadsheet will produce correct values of $x_{\text{max}}$, $y_{\text{max}}$, and $x_{\text{ymax}}$. If you miss the vertex or the horizontal hoop, but come reasonably close, you should try again. There could be enough random variation in the firing of the cannon to produce a miss, even if your calculations and data are correct. However, if you can’t hit it in several tries, or if your calculations appear to be very far off, then you should adjust the hoops by trial and error until you send the ball through both. In that case, record your experimental values of $x_{\text{max}}$, $y_{\text{max}}$, and $x_{\text{ymax}}$, and include them in your report.

1.4.3 Part 3: Hitting the hoop

The final part of this lab requires you to shoot the ball through a hoop at a given height and distance. If you know $v_0$, you should be able to calculate the value of $\theta_0$ necessary to hit a target at $(x_{\text{hoop}}, y_{\text{hoop}})$. Your spreadsheet should be arranged so that all you have to do is enter $x_{\text{hoop}}$ and $y_{\text{hoop}}$, and $\theta_0$ will be calculated automatically. The spreadsheet should also calculate the coordinates of the vertex for that $\theta_0$. Remember that if two different values of $\theta_0$ are possible, you will want to use the larger of them.

You will probably have two different values of $v_0$ from the horizontal and the vertical firing. You won’t be able to interpolate between these values on $\theta_0$, since you’ll need $v_0$.
to calculate $\theta_0$ in the first place. We suggest that you use the value from the vertical firing, since you will be using the larger of the two angles $\theta_0$. However, you should keep this in mind as a possible source of error in your results.

When you are ready to do this part of the lab, tell the attendant. He may do a preliminary test on your spreadsheet, by giving you values of $v_0$, $x_{\text{hoop}}$, and $y_{\text{hoop}}$, then seeing if your spreadsheet comes up with a correct value for $\theta_0$ and with correct coordinates for the vertex.

The attendant will then place a horizontal hoop at some position. You will have to measure the distances $x_{\text{hoop}}$ and $y_{\text{hoop}}$, then calculate $\theta_0$, $x_{\text{ymax}}$, and $y_{\text{max}}$. You should measure $x_{\text{hoop}}$ and $y_{\text{hoop}}$ to the center of the hoop. Since you will try to make the center of the ball pass through that point, you won’t need to take into account the radius of the ball.

The lab attendant may try to trick you by setting the hoop in a position that’s completely out of range of the spring cannon. If this is so, then your spreadsheet should let you know it, and you should inform the attendant that this is the case. If you’re correct, he will then re-position the target so that you can hit it.

You will test your prediction of $\theta_0$ by setting the cannon to that angle and firing it. If your prediction is correct, the ball will pass through the hoop and land in the basket.

If you don’t hit the hoop, but come reasonably close, you should try again. However, if you can’t hit the hoop in repeated tries, or if it appears that your calculation is off by a considerable margin, then you should adjust $\theta_0$ by trial and error until you hit the target hoop. In that case, you should record both your theoretical value $\theta_0$ and the experimental value $\theta_{0,\text{exp}}$.

Once you’ve hit the target hoop, calculate the coordinates of the vertex $(x_{\text{ymax}}, y_{\text{max}})$ for the value of $\theta_0$ that you used (whether calculated or experimental). Place the vertical hoop with its center at that point and fire the ball. It should pass through both hoops. If it doesn’t go through the vertex hoop but comes reasonably close, try again. However, if you can’t hit the vertex hoop in repeated attempts, or if your calculated values appear to be off by a large amount, then you should adjust the vertex hoop by trial and error until you can get the ball through both hoops. In that case, you should record both your theoretical values for the vertex coordinates and the trial-and-error values that actually worked.

### 1.5 Results and analysis

Before you come to the lab, you should construct a spreadsheet into which you can enter your experimental data, and which will calculate the results you need.

Each cell on your spreadsheet should be clearly labelled, whether you’re going to enter data into it or read the results of calculations from it. It’s not a bad idea to write a
set of instructions: say, on a second worksheet. You want someone else to be able to understand and use your spreadsheet; and if you should have to return to it after you haven’t used it in several months, you don’t want to spend a lot of time trying to figure out how it works.

It’s also a good idea to protect the calculation cells on your worksheet, so that you don’t accidentally write over a cell with a formula in it.

Before you start the lab, you should test every part of your spreadsheet very carefully. If you’re working in a group, it’s a good idea for every member of the group to come up with his own set of test data, and to test the spreadsheet independently. This increases the likelihood that someone will spot a problem before it causes trouble in the lab.

The following describes cells that you will have to have in your spreadsheet. You are free to use more if you like, so long as they’re labelled clearly. If you’re evaluating a fairly complicated formula, for instance, you might prefer to work by short steps, producing lots of intermediate results, rather than trying to code the whole complicated formula in a single cell.

To make it easier to test your spreadsheet, it’s a good idea to use intermediate cells whose values you can change without overwriting a complicated formula. For example, suppose that you calculate \( v_0 \) in cell B10, and then need to use that value of \( v_0 \) in a later formula to calculate \( y_{\text{max}} \). Instead of having the \( y_{\text{max}} \) formula call cell B10 directly, you might put the formula “=B10” in cell B11, and have the \( y_{\text{max}} \) formula call cell B11. This might seem like an unnecessary extra step; but it allows you to enter numerical values for \( v_0 \) in cell B11 in order to test your \( y_{\text{max}} \) formula, without overwriting the formula that you’ve entered (and carefully tested) in cell B10.

1.5.1 Part 1: Initial velocity

Your spreadsheet should have cells for \( g \), for \( h_{\text{cm}} \), and for \( d_{\text{ball}} \) and/or \( r_{\text{ball}} \).

You will need cells for the three measurements of \( x_{\text{max}} \) that you make when firing the cannon horizontally. Another cell should calculate the average of these three measurements; and another should contain the value of \( v_0 \) that you calculate from this average \( x_{\text{max}} \).

You will also need cells for the three measurements of \( y_{\text{max}} \) that you make when firing the cannon vertically. Another cell should contain the average of these three measurements; and another should contain the value of \( v_0 \) that you calculate from this average \( y_{\text{max}} \).

1.5.2 Part 2: Fixed angle of elevation

For the second part of the experiment, you will need a cell in which to enter the value of \( \theta_0 \) that you’re given, and another one for the height of the horizontal target hoop. You’ll
need a cell for $v_0$, whether you use one of the two $v_0$ values directly or whether you try to interpolate between them.

You will then need cells to contain your calculated values of $x_{\text{max}}$, $x_{\text{ymax}}$, and $y_{\text{max}}$. You should have additional cells in which to enter the experimental values of these three quantities, in case your calculated values don’t work and you have to resort to trial and error. Your cells should be labelled so that it’s very clear which ones are theoretical calculations and which are experimental measurements.

### 1.5.3 Part 3: Hitting the hoop

You will need cells in which to enter the values of $x_{\text{hoop}}$ and $y_{\text{hoop}}$. You should have a cell for $v_0$, whether you use one of the two experimental values from Part 1 or interpolate in some way between them.

Your calculations will be somewhat long and involved. You will need to solve equation (34) for $\theta_0$, which will probably take several steps. This is a situation in which it’s a good idea to calculate intermediate results. We recommend that you have cells for $a$, $b$, and $c$ from equation (36), and one in which you calculate $\theta_0$ using equation (39). Since these are complicated formulas, it doesn’t hurt to use additional intermediate cells with simpler formulas. Of course, all of your cells should be clearly labelled; in particular, it should be very clear whether a cell contains the result of a calculation, or whether it’s one for data to be entered in.

Remember that the attendant may give you a target that’s impossible to hit. You will want your spreadsheet to give a clear warning in this case. A useful (but not required) approach is to use conditional formatting to change the background color of a cell for an out-of-range target.

You will need a cell for $\theta_{0,\text{exp}}$, in case the value that you calculate doesn’t work. Once you’ve got a value for $\theta_0$ that allows you to hit the horizontal target hoop, you’ll need cells in which to calculate the vertex coordinates $x_{\text{ymax}}$ and $y_{\text{max}}$. In case those calculations don’t work out, you’ll need two more cells for the experimental values of the vertex coordinates that you find by trial and error.

Since these are complicated calculations, you need to test them with particular care before you come to the lab. Before the lab attendant watches you shoot your ball at the hoop, he may test your spreadsheet by giving you values of $v_0$, $x_{\text{hoop}}$, and $y_{\text{hoop}}$; your spreadsheet should come up with the correct value or values for $\theta_0$. 
1.6 Your report

Your lab report should be written according to the guidelines in “Physics 210 Lab Procedures”, and according to the following suggestions.

Introduction

Your introduction should be a very brief account of what you did and what you found. In particular, were you able to put the ball in the basket? Did the apparatus work as expected? Don’t describe the apparatus in detail or lay out the reasoning behind your results; you’ll do that later in the report.

Theory

You don’t need to give the full derivation of the equations used in this lab, I’ve already done that for you. You should give a summary of my derivations. Be sure and include the key steps for each of the three parts: finding the initial velocity, fixed angle firing, and putting the ball in the hoop.

Lab procedure

Describe the experimental setup. You can assume that your reader knows about spring cannons, so you don’t have to describe it in detail and explain how it works. However, you should let the reader know how it was arranged and used so that they could repeat your experiment.

Don’t forget to explain how you got the measurements. For example, your description should include how you made sure the cannon was vertical, horizontal, or at an angle \( \theta_0 \) (including error estimate); how you measured the range and the maximum height of the ball; and how you you set up the vertical and horizontal hoop (basket).

Briefly describe your procedure for putting the ball through the hoop and into the basket. In particular, be sure to explain your procedure for trying to keep the firings consistent.

Do not include your data from these measurements; that goes in the next section. If you had problems during the measurements (for example, if you bumped the vertical hoop, or had trouble getting the ball to go into the basket, describe the problems and what you did about them).

Results and analysis

Summarize your results here. Be sure and break your results into the three parts of this experiment: finding the initial speed of the ball; fixed-angle height and range predictions.
and putting the ball in the basket. Be sure and answer all the “questions to keep in mind” listed below.

This section should also include a discussion of your results. How well did your spreadsheet predict the height and range? Don’t just give your opinion: support it by reference to the data.

You might also wish to discuss how the experiment could have been improved. Consider both your experimental technique and the apparatus itself.

**Conclusion**

This should be a very brief restatement of what you found. Were your ball launches fairly consistent? Once you got the bugs worked out of your spreadsheet and the apparatus working, were you able to consistently put the ball in the hoop? If you had trouble predicting the range, how did you resolve it? If the results weren’t as good as you’d hoped, what could have been done to improve the outcome?

**Things that should be included in your report:**

When writing out a your formal lab report, be sure to include the answers to the following questions.

If your answer to a question is the value of a quantity that you’ve measured or calculated, do not just write the number; tell what quantity it represents, and include units. For example, if you are asked for $x_{\text{max}}$, don’t write “2.33”; write $x_{\text{max}} = 2.33 \text{ m}$. Failure to do this will adversely affect your grade.

**Part 1: Initial velocity**

- What value did you find for $v_0$ when you fired the ball horizontally? Denote this speed by $v_{0,\text{hor}}$.
- What value did you find for $v_0$ when you fired the ball vertically? Denote this speed by $v_{0,\text{vert}}$.
- What was the relative error of the two $v_0$ measurements, relative to the average of the two? The formula for this relative error is

  $$\frac{|v_{0,\text{vert}} - v_{0,\text{hor}}|}{v_{0,\text{vert}} + v_{0,\text{hor}}}$$

  Express your answer as a percentage.

  Notice that this is equal to

  $$\frac{2\delta v_0}{2v_{0,\text{best}}} \quad \text{where} \quad v_{0,\text{best}} = \frac{v_{0,\text{vert}} + v_{0,\text{hor}}}{2}$$
Part 2: Fixed angle of elevation

- When you fired the cannon at a specified $\theta_0$, how did you calculate the value of $v_0$ that you used in calculating $x_{\text{max}}$, $x_{\text{ymax}}$, and $y_{\text{max}}$?

- When you fired the cannon at a specified $\theta_0$, what was your calculated value of $x_{\text{max}}$? Did you hit the horizontal target hoop at that distance? If not, what was your experimental $x_{\text{max}}$, and what was the relative error?

- When you fired the cannon at a specified $\theta_0$, what was your calculated value of $x_{\text{ymax}}$? Did you hit the vertical target hoop at that distance? If not, what was your experimental $x_{\text{ymax}}$, and what was the relative error?

- When you fired the cannon at a specified $\theta_0$, what was your calculated value of $y_{\text{max}}$? Did you hit the vertical target hoop at that height? If not, what was your experimental $y_{\text{max}}$, and what was the relative error?

- In view of your results, did you choose an accurate value for $v_0$? Would a different value of $v_0$ have produced better results? If so, should that different value have been larger or smaller than the value that you used in your calculations? How would that different value have been related to the two experimental values from the horizontal and the vertical firing?

Part 3: Hitting the hoop

- In Part 3 of this lab, how did you calculate the value of $v_0$ that you used in calculating $\theta_0$ given $x_{\text{hoop}}$ and $y_{\text{hoop}}$?

- What value of $\theta_0$ did your spreadsheet yield? What was the actual elevation $\theta_{0,\text{exp}}$ that you used to get the ball through the horizontal hoop? What was the relative error of the predicted angle?

- Once you’d found a value of $\theta_0$ that allowed you to hit the target hoop, what were the calculated coordinates of the vertex ($x_{\text{ymax}}, y_{\text{max}}$) for that $\theta_0$? If the ball didn’t pass through the vertical hoop at these coordinates, what were the experimental values that you found by trial and error? What were the relative errors of $x_{\text{ymax}}$ and $y_{\text{max}}$?

- Based on your results from trying to hit the horizontal target hoop and the vertical vertex hoop, would you modify your estimate of $v_0$? Would a different value of $v_0$ have predicted the experimental values of $\theta_0$, $x_{\text{ymax}}$, and $y_{\text{max}}$? Would this value of $v_0$ have fallen between the two experimental values from the horizontal and the vertical firing?