Compound Interest Primer

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1 Elementary Theory of Interest

Interest is the additional money that a borrower pays a lender for the use of the lender’s money. It is often expressed as a percentage of the sum borrowed. The amount of interest depends, among other things, on the amount of money borrowed, and on the time that it takes for the borrower to repay the loan.

1.1 Defining the variables

Let’s define three variables that we’ll use throughout our study of interest.

We will use $P$ to denote the principal: the amount of money borrowed, or the present value of the money invested. The interest rate will be $r$; and $r$ must be in decimal form for computation—if the interest rate is 15%, then $r = 0.15$. For time, we will use $t$; this will almost always be in years. If the time is not given in years, then it must be converted into years. For example, if $t = 15$ months, then it must be converted to years: $t = 1.25$ years.

The named, or quoted, interest rate $r$ is referred to as the nominal interest rate, or the annual interest rate. This is typically the advertised rate that banks and loan offices use: get a loan at only 8% interest! However, knowing the annual interest rate, in and of itself, holds limited meaning. To compute the actual amount of interest that you will have to pay requires knowing the compounding scheme. That is, to really predict how an investment will grow with time, you must know the pair: the interest rate $r$, and the compounding scheme. Therefore, it is useful to think of these as an order pair:

$$(r, \text{Compounding scheme}).$$

We now give a formal definition of simple and compound interest. The rest of the chapter will be devoted to giving you a deeper understanding of this definition.

Definition 1.1. In **simple interest**, only the original principal earns interest. In **compound interest**, the accumulated interest payments are added to the original principal and earn interest along with it.
1.2 Simple Interest

The formula for simple interest is: \( I = Prt \), where \( I \) represents the amount of interest earned after \( t \) years. If you invest an amount \( P \) at a rate of \( r \) and earn simple interest for \( t \) years, then at the end of that time the value of your account is \( F = P + I \): principal plus interest. Combining all the terms gives the mathematical expression

\[
F = P + I = P + Prt = P(1 + rt).
\]

Example 1.1. Suppose that you want to invest 1000 dollars at 5% interest for 2 years. After one year you earn 5% of 1000 dollars: namely, 50 dollars. You withdraw the interest and bury it in your backyard, leaving only the original 1000 dollars to earn interest in the second year. At the end of the second year you earn another 50 dollars. At the end of 2 years, you’ve earned a total of \( 2 \times 50 \) dollars in interest. Here’s a time map of your earnings

\[
1000 \mapsto 1000 + .05(1000) \quad \text{(at the end of year 1)}
\]

\[
\mapsto 1000 + .05(1000) \quad \text{(first year earnings)} + .05(1000) \quad \text{(second year earnings)}
\]

\[
= 1000 + 2(.05(1000)) = 1000(1 + 2(.05)) \quad \text{(total earnings at the end of year 2)}
\]

The general formula for simple interest is

\[
F = P(1 + rt),
\]

(1.1)

where \( P \) = initial principal, \( F \) = future value, \( r \) = annual interest rate, and \( t \) = number of years. Typically, \( P \) and \( r \) are known, and the future value is a function of time (measured in years). In our example, you invest $1000 at a simple interest rate of 5%. Then the future value as a function of time is \( F(t) = 1000(1 + .05t) \). This is a linear function, since it fits the form \( y = ax + b \). In this example, the domain was \( t = 0, 1, 2, \ldots \) However, we can extend the formula to remain valid for all time \( t \in \mathbb{R} \) by simply agreeing that for any time in between \( t = 0, 1, 2, \ldots \) we will take the value of the investment to be worth the amount \( F \) given by the equation (1.1).

Note: This assumes that interest is paid uniformly over any fractional period of time. Many accounts only pay interest in discrete payments at the end of fixed time periods. You only get interest for full periods; if you close the account partway through a period, you don’t get any interest for that fraction of a period. Suppose, for example, that a bank pays interest quarterly (with a period of 1/4 year). The value of the account is shown on the graph:
Example 1.2. You borrow $10,000.00 on January 1, 2000; the simple interest rate is 7%.
(a) What amount must be repaid on January 1, 2004?
(b) On that date, what is the total interest on the loan?

Solution: (a) Given $P = 10,000$, $r = 0.07$, and $t = 4$, we have

$$F(4) = P(1 + rt) = 10,000(1 + (0.07)(4)) = 12,800.$$ 

(b) $I = F - P = 12,800 - 10,000 = 2800$.

We can also ask: “How much money should be invested today in order to have a certain amount at a given future date?”

Example 1.3. How much should an investor deposit in the bank now in order to have $1000 in the bank half a year from now? The simple interest rate is 8%.

Solution: Given $F = 1000$ (this is the future value that we want), $r = 0.08$, and $t = 1/2$, we want to find $P$. Solving $F = P(1 + rt)$ for $P$ gives

$$P = \frac{F}{1 + rt} = \frac{1000}{1 + (0.08)(1/2)} = \frac{1000}{1 + 4/100} = \frac{1000}{1.04} \approx 961.54.$$
1.3 Compound Interest

1.3.1 Annually Compounded Interest

Again, let \( P \) denote the principal (initial investment), and \( r \) denote the annual interest rate. Suppose the interest is compounded annually: at the end of every year, that year’s interest is added to the principal to earn interest in future years.

After one year,

\[
F = P(1 + r).
\]

After two years,

\[
F = [P(1 + r)](1 + r) = P(1 + r)^2.
\]

After three years,

\[
F = [P(1 + r)^2](1 + r) = P(1 + r)^3.
\]

Continuing in this way, after \( t \) years,

\[
F = P(1 + r)^t,
\]

where \( t = 0, 1, 2, \ldots \). Notice that over each compounding period, the interest is determined from the simple interest formula applied to the principal over the time period in between the compoundings. Thus, the building block for the annual compound interest formula is the simple interest formula.

Although we have only derived the formula for integer values of the domain \( t = 0, 1, 2, \ldots \), we can extend the formula to be valid for all values of time \( t \). Thus, this is valid over any time period, including a fractional time period. How can we do this? We simply define it to be valid for all time \( t \). This should not be too surprising, since the formula is valid for all \( t \)-values, and so long as all of the investors all agree, we can take the value of the formula to be the value of the investment. But this formula is far from random. It has the built-in structure that it compounds your money over evenly spaced time periods—in the case of annual interest, a year.

**Example 1.4.** Going back to example (1.1), suppose we had not taken the 50 dollars in interest out at the end of the first year. Then over the second year, those 50 dollars would have earned interest along with the 1000-dollar initial principal.

\[
1000 \rightarrow 1000 + .05(1000) = 1000 + 50 \quad \text{(at the end of year 1)}
\]
\[
\rightarrow (1000 + 50) + .05(1000 + 50)
\]
\[
= 1000(1 + .05)(1 + .05) = 1000(1 + .05)^2 \quad \text{(at the end of year 2)}
\]
The general formula for annually compounded interest is

\[ F = P(1 + r)^t. \]

Typically \( P \) and \( r \) are known, and the future value is a function of time (measured in years). Paralleling the previous example, suppose you want to invest $1000 at 5% compounded annually. Then the future value as a function of time is \( F(t) = 1000(1 + .05)^t \). This is a not a linear function, since it does not fits the form \( y = ax + b \). By default, it is a nonlinear function.

**Example 1.5.** (Annually Compounded Interest) Invest $1 at 10% interest, compounded annually, over 30 years.

**Solution:** Given: \( P = 1 \), \( r = 0.1 \), and \( t = 30 \). Find \( F \).

\[
F = P(1 + r)^t \\
= (1.1)^{30} \\
= 17.4494
\]

The net increase was roughly 1645%! Now suppose that instead of investing $1, you had invested $10,000. What would your investment be worth? Since the relationship between \( F \) and \( P \) is linear, we can just multiply our previous result by 10,000 to get \( F = 10,000(17.4494) = 174,494 \) to the nearest dollar.

### 1.3.2 Compounding Intervals

Interest can be compounded more often than once per year. New interest might be added to the old principal at the end of every quarter, or every month, or even weekly or daily.

Let \( n \) denote the number of times per year that interest is compounded (number of conversion periods in one year), \( P \) denote the principal (initial investment), and \( r \) denote the annual interest rate (assumed constant). For example if \( n = 2 \), then the interest is compounded twice a year. Below is a list of common discrete compounding periods and their corresponding names.

- \( n = 1 \) compounded annually
- \( n = 2 \) compounded semi-annually
- \( n = 4 \) compounded quarterly
- \( n = 12 \) compounded monthly
- \( n = 52 \) compounded weekly
- \( n = 365 \) compounded daily
Suppose the interest is compounded \( n \) times a year: at the end of every period of time \( \frac{1}{n} \) years, that period’s interest is added to the principal to earn interest in future periods. Thus we have divided the year into \( n \) intervals, each with a duration of \( \frac{1}{n} \) years. We then apply the simple interest formula to the principal over each subinterval (period). The interest earned over one period is \( I = Prt = P \frac{r}{n} = P \frac{r}{n} \).

After one period (\( t = \frac{1}{n} \) years),

\[
F = P \left( 1 + r \cdot \frac{1}{n} \right) = P \left( 1 + \frac{r}{n} \right).
\]

After two periods,

\[
F = \left[ P \left( 1 + \frac{r}{n} \right) \right] \left( 1 + \frac{r}{n} \right) = P \left( 1 + \frac{r}{n} \right)^2.
\]

After three periods,

\[
F = \left[ P \left( 1 + \frac{r}{n} \right)^2 \right] \left( 1 + \frac{r}{n} \right) = P \left( 1 + \frac{r}{n} \right)^3.
\]

After \( n \) periods (one year), the principal is compounded \( n \) times. The final value is then

\[
F = P \left( 1 + \frac{r}{n} \right) \left( 1 + \frac{r}{n} \right) \ldots \left( 1 + \frac{r}{n} \right) \quad \text{(\( n \) terms)}
= P \left( 1 + \frac{r}{n} \right)^n.
\]

After two years, the final value is

\[
F = \left[ P \left( 1 + \frac{r}{n} \right)^n \right] \left( 1 + \frac{r}{n} \right)^n = P \left( 1 + \frac{r}{n} \right)^{2n}.
\]

After \( t \) years, the final value is

\[
F = P \left( 1 + \frac{r}{n} \right)^{nt},
\]

where \( t \) need not be an integer. Again, we have derived the formula for values of \( t = \frac{m}{n} \), where \( m, n \in \mathbb{N} \), and we extend the formula, via definition, to be valid for all time \( t \).

**WARNING:** \( t \) must always be given in years.
Example 1.6. (Weekly Compounded Interest) Invest $10,000 at 10% interest, compounded weekly, over 1 and 30 years.

Solution: Given: \( P = 10,000 \), \( r = 0.1 \), \( n = 52 \), and \( t = 1, 30 \). Find \( F \).

After one year (\( t = 1 \)),

\[
F = 10,000 \left( 1 + \frac{0.1}{52} \right)^{52 \cdot 1} = 11,050.65
\]

After thirty years (\( t = 30 \)),

\[
F = 10,000 \left( 1 + \frac{0.1}{52} \right)^{52 \cdot 30} = 200,277.6
\]

Compare this last result with the result in example 1.5. That example had the identical scenario, except that the interest was compounded annually instead of weekly.

In summary, the only difference between compounding \( n \) times a year and once a year is that for \( n > 1 \), the compounding is occurring more frequently. Just as with annual interest the simple interest formula is applied to the principal over each of the \( n \) compounding periods. Thus, just as in the case of annual interest, the simple interest formula is a fundamental building block for the discrete compound interest formula.

The frequency of compounding plays an important role in the growth of the principal. Given a fixed principal \( P \), a fixed annual interest rate \( r \), and a fixed time interval \( t \), we can think of the future value \( F \) in equation (1.2) as a function of \( n \), the number of compounding periods per year. The rate at which your investment grows depends on how many times per year the interest is compounded. It turns out that the more compounding per year that the investment is subjected to, the greater the rate of return on the investment. But as we shall soon see, there is a limit on this rate of return.
1.3.3 The limit on the growth rate of an investment as a function of $n$

Using Calculus it can be shown that if $n, m \in \mathbb{N}$ and $m < n$, then

$$
\left(1 + \frac{1}{m}\right)^m < \left(1 + \frac{1}{n}\right)^n.
$$

This inequality proves that

$$f(m) = \left(1 + \frac{1}{m}\right)^m$$

is an increasing function of $m$. Below are four graphs of $f(m)$ confirming this claim. In the top two graphs we plot $f(m)$ as a discrete function of $m$, where $m = 1, 2, 3, \ldots, 10$ (upper leftmost graph), and $m = 1, 2, \ldots, 100$ (upper rightmost graph). In the bottom two graphs, we plot $f(m)$ as a continuous function of $m$ over the intervals $[0, 10]$, $[0, 100]$ respectively. Notice that in all of the pictures, the graphs are strictly increasing.

Figure 1: Notice that as $m \to \infty$, the graphs all seem to be strictly increasing and approaching the same limit. The limit is $e \approx 2.71$.

Let $m = n/r$ and fixing $r > 0$, $P > 0$, and $t > 0$. It follows immediately that

$$F(n) = P \left(1 + \frac{r}{n}\right)^{nt}$$

is an increasing function of $n$ (the number of compounding periods). This is verified in the graph below. We have plotted the future value of $P = 1$ dollar invested at $r = .05$
over a time period of 100 years for the compounding periods $n = 1, 4, 12$. We have also included the graph of simple and continuous compound interest for purposes of comparison.

![Graph of Future Value](image1)

**Figure 2:** Various outcomes for the future value of $1 at a 5\%$ annual interest rate $r$. The simple interest is the linear graph that exhibits the least amount of growth. The graphs of the discrete compounding with $n = 1, 4$ are barely distinguishable even over a one hundred year period of growth. The difference between the $n = 12$ graph and continuous compounding graph cannot be resolved on this vertical scale over the ”short” time period of 100 years!

![Graph Closeup](image2)

**Figure 3:** A closeup of the graphs corresponding to $n = 1, 4, 12, 365$ and continuous compounding. Even in this extreme closeup, the graphs of continuous compound interest and the discrete compound interest with $n = 365$ are indistinguishable!

When graphs become so close together it is sometimes useful to plot the final values in a table for an easier comparison. Below is a table that gives the future value of $P = 1$ dollar that is invested at an annual interest rate of $r = 5\%$ for $t = 100$ years as a function of the rate of compounding.
<table>
<thead>
<tr>
<th>$n$</th>
<th>simple</th>
<th>1</th>
<th>4</th>
<th>12</th>
<th>365</th>
<th>continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(n)$</td>
<td>6</td>
<td>131.50</td>
<td>143.88</td>
<td>146.88</td>
<td>148.36</td>
<td>148.41</td>
</tr>
</tbody>
</table>

Table 1: A comparison of investment growth as a function of compounding rates. Notice how close $F(365)$ is to $F(\text{continuous})$.

### 1.3.4 Continuous Compounding

For $n > 1$, the final value $F$ is larger than it would be for simple interest at the same annual interest rate $r$. In fact, equation (1.2) is an increasing function of $n$. Hence, more frequent compounding leads to increased yield. While this fact is difficult to show mathematically, intuitively, the reason for it is simple: the sooner you start earning interest on interest, the faster your investment grows. This increment has a limit as the number of times of compounding (per year) increases indefinitely. For a fixed annual interest rate $r$, as $n \to \infty$, the factor

$$
\left(1 + \frac{r}{n}\right)^n
$$

increases to the limit $e^r$. Thus as $n$ gets very large, the compound interest formula becomes

$$
F = Pe^{rt},
$$

(1.3)

where $e := 2.7182818\ldots$ is the exponential growth constant. Here we have used the fact that

$$
\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e,
$$

where $m = n/r$. In the homework you are asked to verify this using MS Excel.

**Advanced Comment 1:** To prove this, let $x = 1/m$ and notice that

$$
\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = \lim_{x \to 0^+} (1 + x)^{\frac{1}{x}},
$$

let $x = 1/m$

$$(1 + x)^{\frac{1}{x}} = \exp \left[ \frac{\ln(1 + x)}{x} \right],$$

and

$$
\frac{\ln(1 + x)}{x} \to 1 \quad \text{as} \quad x \to 0.
$$

The result now follows.

**Advanced Comment 2:** Notice that in both the simple and compound interest formulas, there is a linear relationship between $F$ and $P$. However, in the case of simple
interest, the relationship between $F$ and $r$ and $F$ and $n$ is linear. In the case of compound interest, the relationship between $F$ and $r$ and $F$ and $n$ is nonlinear. This results from the fact that with compound interest, you earn interest on interest, which is not the case with simple interest. The result is that your investment under compound interest grows exponentially rather than linearly. The downside of this is that your credit card debt also grows exponentially rather than linearly.

Example 1.7. (Continuous Compounded Interest) Invest $10,000 at 10% interest, compounded weekly, over 1 and 30 years.

Solution: Given: $P = 10,000, r = 0.1$ and $t = 1, 30$. Find $F$.

After one year ($t = 1$),

$$F = 10,000e^{(0.1)1} = 11,051.71$$

After thirty years ($t = 30$),

$$F = 10,000e^{(0.1)30} = 200,855.4$$

Compare this last result with the result in example 1.6. That example had the identical scenario, except that the interest was compounded weekly instead of continuously.
1.4 The Connection between Growth Rates and Compounding

Compound interest is like a snowball rolling down a hill. The faster the snowball rolls, the more revolutions per unit of time it goes through, hence the more snow it picks up, hence the faster it grows. Similarly, the faster you overturn the interest, the more compounding periods per year, the more money you accumulate. There is limit as to how many revolutions per unit of time that a snowball can obtain without centripetal forces tearing it a part. Just as with the snowball, there is limit to the rate of growth under compounding. It is the rate corresponding to continuous compounding.

We have seen that if $P$, $r$, and $t$ are fixed, then simple interest has the slowest growth rate. This growth rate is linear. Next, we looked at compound interest that exhibits nonlinear growth. We found that the more compounding periods per year, the faster the investment grows (it is an increasing function of $n$). This increased growth reaches a plateau in the limit as $n \to \infty$. It is known as continuous compounding. We can now paint a ‘mathematical picture” of the growth rates. Let $P$, $r$, and $t$ be fixed. We then have the following inequality relations between the various growth rates as a function of compounding periods:

$$P(1 + rt) < P \left(1 + \frac{r}{n}\right)^{nt} < P \left(1 + \frac{r}{m}\right)^{mt} < Pe^{rt},$$

where $n, m \in \mathbb{N}$ and $n < m$. For example, the return on an investment of $P$ dollars at $r\%$ annual interest over $t$ years will grow at various rates depending on the compounding period. The inequality between the returns that result from simple, discrete compounding with $n = 1, 4, 12, 365$ and continuous compounding on these investments gives a natural ordering between the various investment schemes, as shown below

$$P(1 + rt) < P(1 + r)^{t} < P \left(1 + \frac{r}{4}\right)^{4t} < P \left(1 + \frac{r}{12}\right)^{12t} < P \left(1 + \frac{r}{365}\right)^{365t} < Pe^{rt}. $$

This inequality is confirmed in table 1 for $P = 1, r = .05$, and $t = 100$. From these inequalities and from figures 2 and 3, it should now be clear that simple and continuous compound interest act like an envelope for discrete compounding, with the simple-interest graph being the lower bound for all $n$ and the continuous-compound-interest graph being the upper bound for all $n$. Thus, for fixed $P$, $r$, and $t$, the simple interest graph exhibits the least amount of growth. It is below all the discrete compounding graphs. The smallest discrete compounding graph is the annually discrete compounded interest corresponding to $n = 1$, next the semi-annually discrete compounded interest corresponding to $n = 2$, next the quarterly discrete compounded interest corresponding to $n = 4$, next the monthly $n = 12$, then the yearly $n = 365$, and finally the continuous compound interest graph that sits above all of the discrete compounding graphs.
1.4.1 Comparing Extremes: Simple and Continuous Compound Interest

Below is a graph that shows the difference between the growth rate for simple and compound interest over long time periods. We have chosen to compare the future value formula for simple interest, the slowest-growing interest scheme, to the future value formula for continuous compound interest, the fastest-growing interest scheme, to emphasize the maximum disparity in the rate of growth of an investment under the two schemes. While the difference in the future value of one dollar between the two schemes is quite striking over long times, notice that the graphs of the two functions are pretty close for about the first 20 years. This a little misleading since the principal is only one dollar. With larger principals you would see a significant difference over shorter time scales. However, this too is misleading. No matter what the value of $P$ and $r$, the two graphs would be approximately equal over some finite time interval. Moreover, if we examine the earnings ratio $F/P$ then we would find that the two graphs would be close for $rt$ small (much less than 1). It is this fact that we shall further explore in subsection 1.6.

![Graph showing comparison between simple and continuous compound interest](image-url)

Figure 4: A comparison between the investment of one dollar under simple interest (linear growth) and the investment of one dollar under continuous compound interest (exponential growth). The interest rate is 5%, the time is 100 years.
1.5 An Application of Growth Rates and Compounding

It is now time to put our knowledge about growth rates as a function of compounding to use. Understanding of the connection between rate of growth and the rate of compounding can give you quick-and-dirty approximate answers for multiple-choice test questions.

Suppose you’re taking a business college entrance exam. The exam consists of 20 problems with a time limit of 40 minutes. You are only given a 4-button calculator. You can only spend an average of two minutes on each problem. How can you quickly determine the correct solutions to problems involving compound interest?

By a 4-button calculator, we mean one that can only add, subtract, multiply, and divide. There are no function buttons like the square root button or the exponential button. Let us digress to discuss some examples of calculations that you can and can’t do with a 4-button calculator.

With a 4-button calculator, you can easily compute \( F = P(1 + rt) \) given \( P \), \( r \), and \( t \). To compute \( rt \), we multiply; to compute \( 1 + rt \), we add; and finally, to compute \( P(1 + rt) \), we multiply again. Thus the simple-interest formula is 4-button calculator friendly.

**Example 1.8.** What is the future value of $100,000 invested at a simple interest rate of 4%, after 5 years? Round your answer to whole dollars? Your choices are

(a) $100,000
(b) $95,000
(c) $120,000
(d) $122,140

**Solution:** \( F = 100,000(1 + .04)5 = 120,000 \). This calculation was straightforward. The answer is (c).

It’s worth pointing out that you could compute a term of the form \( x^n \), if \( n \) is an integer. Recall the definition of the integer exponent is

\[ x^n = x \cdot x \cdots x \quad (x \text{ is multiplied by itself } n \text{ times}). \]

For example, \( 5^3 = 5 \cdot 5 \cdot 5 \).

On the other hand, you can’t calculate square roots like \( 5^{1/2} \). Remember, you don’t have a square root button. How do we multiply 5 by itself 1/2 times? What would this interpretation of a rational exponent even mean? It is worth pointing out that there are mathematical algorithms for computing the square root of a number to any desired accuracy, but these are beyond the scope of our discussion.
Another limitation is that you cannot directly compute $e^{rt}$ on a 4-button calculator, because you can only add, subtract, multiply, and divide. Both the square root button and the exponential button on your calculator use advanced mathematical algorithms to compute $\sqrt{x}$ and $e^x$ for a given value of $x$, and chances are that you don’t know these algorithms. Moreover, you don’t have time to carry out a lengthy calculation, even if you do know how to do it. You only have two minutes per problem!

**Example 1.9.** What is the future value of $100,000$ invested at a continuously compounded annual rate of 4%, after 5 years? Round your answer to whole dollars? Your choices are

(a) $100,000  
(b) $95,000  
(c) $120,000  
(d) $122,140

**Solution:** Let’s start by trying a direct approach. We’re given $P$, $r$, and $t$, and we’ll try substituting them into the formula for continuous compound interest. Using only $\{+,-,\times,\div\}$, we can get as far as $F = Pe^{rt} = 100,000e^{(.04)5} = 100,000e^{.2}$. Without a key for $e$, we could try to approximate $e$ by $2.718$ and substitute this value into the expression to get $100,000 \cdot (2.718)^{0.2}$. O.K., now what? How are you going to compute $(2.718)^{0.2}$ using only $\{+,-,\times,\div\}$? Notice that the exponent is 0.2, not 2. We’ve hit a dead end.

The correct approach to this problem is to first notice that we cannot determine the exact solution using a 4-button calculator. We must find some indirect way to find the solution. What qualitative properties do we know about continuous compound interest? One thing that we have learned in this section is that for a fixed annual interest rate $r$, a fixed principal $P$, and a fixed time $t$, the growth rates are increasing functions of $n$. In particular, an investment earning 4% interest, compounded continuously, grows much faster than an investment earning 4% simple interest. The future value of $P = $100,000 invested at a simple interest rate of $r = .04$ over $t = 5$ years is a quantity that we can compute. In fact, we’ve already computed it in example 1.8. It was found to be $F = $120,000. Looking at our choices (a)-(d) we see that it must be (d) since this is the only value greater than the future value under a simple interest rate.

It is worth mentioning that, without doing any calculations at all, we can immediately rule out possibilities (a) and (b). Choice (a) is the value of the principal, and choice (b) is less than the principal.
1.6 Approximating Compound Interest by Simple interest

The following subsection is advanced material. This material maybe skipped without jeopardizing your understanding of future material.

We are now going to explore the conditions under which compound interest can be approximated by simple interest. You should be very interested in this subject, since simple interest is 4-button calculator friendly.

Referring to figure 4, we see that we could approximate the future value of the $1 investment, as a function of time, under continuous compound interest by the simple interest investment for times less than 20 years. To gain a deeper understanding of this fact, let us examine the two formulas side by side. Since the principal $P$ is just a scaling factor, we will examine the resulting equations for the earnings ratio $F/P$ instead of just $F$. Moreover, the earnings ratio is a more accurate measure of the growth rate.

\[
\frac{F}{P} = e^{rt} \text{ (continuous compound interest)} \quad \frac{F}{P} = 1 + rt \text{ (simple interest)}
\]

Let $x = rt$

\[
\rightarrow \frac{F}{P} = e^x \text{ (continuous compound interest)} \quad \frac{F}{P} = 1 + x \text{ (simple interest)}
\]

It is clear that we can approximate continuous compound interest by simple interest, with a small relative error (defined as $|F - P|/P$), so long as we can approximate $e^x$ by $1 + x$. Let us start by examining the graphs of these two functions.

![Figure 5: A comparison between the graph of $y = e^x$ and $y = 1 + x$](image)

Notice that the graphs are approximately equal from $x = 0$ to about $x = 0.2$. In practical terms, this means that we can use the simple interest formula for $F/P$ to approximate the continuous compound interest formula for $F/P$ over the interval $0 \leq x \leq .2$. 
Since \( x = rt \), we have \( 0 \leq rt \leq .2 \) as the region of validity for the approximation. Thus the larger \( r \) is, the smaller the time interval over which the approximation is valid. Conversely, the smaller \( r \) is, the larger the time interval over which the approximation is valid.

### 1.6.1 Advanced Material: Introduction to Taylor Series

We can also approximate both discrete and continuous compound interest by the simple interest formula algebraically. Let’s start with the discrete compounding case first. If we compound the interest \( n \) times in one year, then over the first interval \( [0, \frac{1}{n}] \) the discrete formula \( F = P(1+r/n)^{nt} = P(1+r/n) \), which is the simple interest formula. The results are exact. Over the first and second interval \( [0, \frac{2}{n}] \), the investment has grown to

\[
F = P \left( 1 + \frac{r}{n} \right)^2 = P \left( 1 + \frac{r}{n} \right) \left( 1 + \frac{r}{n} \right) = P \left( 1 + 2 \frac{r}{n} + \left( \frac{r}{n} \right)^2 \right) \approx P \left( 1 + \frac{2r}{n} - \frac{r^2}{n^2} \right),
\]

where you should recognize the last expression as the simple interest formula with \( t = 2/n \). The approximation is valid so long as \( P(r/n)^2 \) is small. Now if \( r/n \) is small (much less than one), then \( (r/n)^2 \) is even smaller. Since most real-world \( r \)'s are between 1% and 18%, so \( .01 < r < .18 \), for \( n \geq 1 \) we see that the ratio \( r/n \) will be small, hence the approximation will be valid provided that \( P \) is not so large as to off-set the smallness of \( (r/n)^2 \).

**Question:** What do we mean by small error? That is, can the difference of two values be small if it is greater than one?

**Answer:** Two values can be “close” even if the error between them is much greater than one! It all depends on what we mean by close.

Keeping the initial principal in the approximation can lead to unnecessary complication. To simplify the analysis we will only focus on approximating the earnings ratio \( F/P \). While this leads to certain limitations, it has the advantage of simplicity. Moreover, the error that is usually most relevant between simple interest and compound interest is the relative error, not the absolute error. If we let \( F_1(n) = P(1+r/n)^{nt} \) and \( F_2 = P(1+rt) \), then the relative error between these two functions is \((F_1 - F_2)/F_2 \). For example, if the error in approximation is \$1000 is this a large error, or a small error? It really depends on the size of the investments that are being considered. If we are investing \( P = 1,000,000 \) dollars, then the error is small. If we are investing \( P = 1,000 \), then the error is large.

The relative error can be expressed in terms of the relative earnings.

\[
\frac{F_1 - F_2}{F_2} = \frac{\frac{F_1}{P} - \frac{F_2}{P}}{\frac{F_2}{P}}.
\]

Thus, it suffices to work with earnings ratio \( F/P \) in place of \( F \).
Let us approximate the earnings ratio over the interval $[0, \frac{3}{n}]$.

\[
\frac{F}{P} = \left(1 + \frac{r}{n}\right)^3
= \left(1 + \frac{r}{n}\right) \left(1 + \frac{r}{n}\right) \left(1 + \frac{r}{n}\right)
= \left(1 + 3\frac{r}{n} + 3\left(\frac{r}{n}\right)^2 + \left(\frac{r}{n}\right)^3\right) \approx \left(1 + r\frac{3}{n}\right),
\]

The approximation is valid so long as $3\left(\frac{r}{n}\right)^2 + \left(\frac{r}{n}\right)^3$ is small. The smallness of this expression is related to the smallness of $r/n$.

The next obvious question is: over what time intervals is this approximation valid? If we looked over longer and longer time intervals, then the approximation of compound interest by simple interest must break down. This fact is evident in the graph 4. To answer this question for discrete and compound interest we will need the use of a mathematical approximation tool known as a Taylor series.

It is a well-known fact in mathematics that certain functions $f(x)$ can be expressed as an infinite series of the form

\[f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.\]

Such a series is referred to as a Taylor series about $x = 0$. This is a special case of a more general series known as power series. A few comments are in order:

1. The above formula may not be valid for every $x$ in the domain of $f$.
2. There is a formula for the $a_n$ that depend on certain properties of the function $f$ that are beyond the scope of this book.
3. In practice you do not sum all of the terms, of which there are infinitely many. Instead, you approximate the function by only a few terms in the series. The sum of these few terms is a polynomial. There is an amazing theorem in mathematics, known as Weierstrass’s Approximation Theorem, that says that any continuous function on a closed interval $[a, b]$ can be uniformly approximated, with arbitrary accuracy, by a polynomial. Given this piece of information, it should not be too surprising that certain continuous functions can be approximated by the first few sums of their Taylor series, since this sum is a polynomial.

Accepting these claims as true, we will now introduce two Taylor series approximations about $x = 0$ for two functions that you are familiar with.

The Taylor series for the first function that we wish to approximate is given by

\[(1 + x)^\alpha = 1 + \alpha x + \frac{1}{2} \alpha (1 - \alpha) x^2 + \cdots. \quad (1.5)\]
This series is valid for $|x| < 1$. If we let $x = \frac{r}{n}$ and $\alpha = nt$, and assume that $\frac{r}{n}$ is small, then we can apply this approximation to show that

$$
\left(1 + \frac{r}{n}\right)^{nt} \approx (1 + rt),
$$

from which it follows that

$$
F = P \left(1 + \frac{r}{n}\right)^{nt} \approx P(1 + rt).
$$

This says that for $\frac{r}{n}$ small enough and for fixed time, we can approximate the discrete compound interest formula by the simple interest formula. **Warning:** this approximation assumes that $\alpha = nt$ is fixed. It is clearly not valid for all time, since $F = P \left(1 + \frac{r}{n}\right)^{nt}$ grows exponentially, whereas $F = P(1 + rt)$ grows linearly.

The Taylor series for the second function that we wish to approximate is given by

$$
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots,
$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ is known as $n$-factorial. Unlike the first series, this series is valid for all values of $x$. However, for large values of $x$ you must add many terms from the series in order for the series to converge to the correct value of $e^x$. But we are not interested in evaluating the series for large $x$. We want to just use the first couple of terms of the series to approximate $e^x$. This means that the truncated series will only be valid for small values of $x$.

It is a mathematical fact that the first few terms of this series approximate $e^x$ well for $|x|$ much smaller than one. If we let $x = rt$ then using (1.7) and keeping only the first two terms in the approximation we arrive:

$$
e^{rt} \approx 1 + rt.
$$

We will find the approximations in (1.6) and (1.8) very useful in approximating the effective annual yield in section 1.7.5.

Perhaps there is no better way to build one’s intuition than to see Taylor series approximations in action. We now show two sets of graphs that compare the exact solution, in this case the graph of $y = e^x$, to a series of approximations: $y_1 = 1 + x$, $y_2 = 1 + x + \frac{1}{2}x^2$, $y_3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$, $y_4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$, and $y_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$. The first graph shows the series of approximations over the interval [0,1], and the second graph shows the series of approximations over the interval [0,5]. The Taylor series about $x = 0$ means that any truncated series, such as $y_1$ through $y_5$, will be most accurate for $x$ near zero. In particular, this means that the approximations should be more accurate on the interval [0,1] than on the interval [0,5].
1.7 Applications Involving the Interest Formulas

We can compute any term, or variable, in the compound interest formulas in terms of the others. We now carry out an exhaustive derivation of each possibility with examples.

1.7.1 Solving for the present value $P$

If we know the final value $F$, the interest rate $r$, the amount of time $t$, and the compounding frequency $n$, then we can determine the principal/present value $P$.

**Discrete compounding:** Start with equation (1.2). To solve for $P$, divide both sides of the equation by $(1 + \frac{r}{n})^{nt}$

$$F = P \left(1 + \frac{r}{n}\right)^{nt} \implies P = \frac{F}{\left(1 + \frac{r}{n}\right)^{nt}}$$

or

$$P = F \left(1 + \frac{r}{n}\right)^{-nt}$$

**Continuous compounding:** Start with equation (1.3). To solve for $P$, divide both sides of the equation by $e^{rt}$

$$F = Pe^{rt}$$

$$\implies P = F e^{-rt}$$
Example 1.10. (Solving for $P$) Given:

$F = \$1000, \quad r = 0.05, \quad n = 365, \quad t = 10$

What is the principal investment required to get $\$1000$ in this case?

Solution:

$P = \$1000 \left( 1 + \frac{0.05}{365} \right)^{-365 \cdot 10} = \$607 \quad \text{(to the nearest dollar)}$

1.7.2 Solving for the Annual Interest Rate $r$

**Discrete compounding:** Start with equation (1.2):

$$F = P \left( 1 + \frac{r}{n} \right)^{nt}.$$  

Next, divide by $P$, and raise both sides of the equation to the power $1/(nt)$

$$\left( \frac{F}{P} \right)^{\frac{1}{nt}} = 1 + \frac{r}{n}.$$  

Lastly, solve for $r$

$$r = n \left[ \left( \frac{F}{P} \right)^{\frac{1}{nt}} - 1 \right].$$

From here forward we will sometimes use a shorthand notation. We use a right pointing arrow $\rightarrow$ to denote “next step”, and over the right arrow will be the operation that we perform on both sides of the equation. For example,

$$F = Pe^{rt} \quad \xrightarrow{\div P} \quad \frac{F}{P} = e^{rt}$$

means that in this step we divide both sides of the equation by $P$.

$$\ln(\cdot) \quad \xrightarrow{\ln} \quad \ln \left( \frac{F}{P} \right) = \ln(e^{rt}) = rt$$

means that in this step we take the logarithm of both sides of the equation. The rest should be self-explanatory. The goal of using this notation is to make the steps seem less wordy, and the presentation more clear.
Continuous compounding: Start with equation (1.3).

\[ F = Pe^{rt} \]

\[ \frac{F}{P} = e^{rt} \]

\[ \ln\left(\frac{F}{P}\right) = \ln(e^{rt}) = rt \]

\[ r = \frac{\ln\left(\frac{F}{P}\right)}{t} \tag{1.9} \]

Notice that if we had attempted to use the natural logarithm to solve for \( r \) in the case of discrete compounding, then we would be stuck. This is because in formula (1.2), the \( r \) is not in the exponent.

**Example 1.11.** (Doubling an investment) An investment of \( P \) dollars doubled in six years under continuous compounding. Assuming the interest rate was constant over this time, what was the interest rate?

**Solution:** Set \( F = 2P \). Then equation (1.3) becomes

\[ 2P = Pe^{rt} \]

\[ 2 = e^{rt} \]

\[ \ln 2 = \ln(e^{rt}) = rt = 6r \]

\[ r = \frac{\ln 2}{6} = 0.1155 = 11.55\% \]

Notice that the value \( P \) of the initial investment dropped out of the equation. Thus the result is valid for any initial investment \( P \).

**Example 1.12.** Suppose $100,000 is invested at a certain rate compounded quarterly, and becomes $150,000 after 30 years. What was the interest rate?

**Solution:** Given:

\( F = 150,000, \ P = 100,000, \ n = 4, \ t = 30. \)

\[ F = P\left(1 + \frac{r}{n}\right)^{nt} \]

\$150,000 = $100,000 \left(1 + \frac{r}{4}\right)^{4 \cdot 30} \]

\[ \frac{3}{2} = \left(1 + \frac{r}{4}\right)^{120} \]
Now, take the 120th root of both sides.

\[
\left( \frac{3}{2} \right)^{\frac{1}{120}} = 1 + \frac{r}{4}
\]

\[r = 4 \left( 1.5^{\frac{1}{120}} - 1 \right) = 0.013538... = 1.35\%
\]

### 1.7.3 Solving for \( t \)

**Discrete compounding:** Start with equation (1.2).

\[F = P \left( 1 + \frac{r}{n} \right)^{nt} \]

Divide both sides of the equation by \( P \) and take the natural log of both sides of the equation (since the variable we wish to solve for is in the exponent). This yields

\[
\ln \left( \frac{F}{P} \right) = \ln \left[ \left( 1 + \frac{r}{n} \right)^{nt} \right].
\]

Next, using the property of logarithms \( \ln(a^b) = b \ln(a) \), we arrive at the solution

\[
t = \frac{\ln \left( \frac{F}{P} \right)}{n \ln \left( 1 + \frac{r}{n} \right)}. \quad (1.10)
\]

**Example 1.13.** (Increasing an investment \( m \)-fold: discrete case) We can use the above formula to answer the question: How long does it take to increase an investment from \( P \) dollars into \( F = mP \) dollars at an interest rate of \( r\% \)?

**Solution:** Setting \( F = mP \) in the above formula yields

\[
t = \frac{\ln m}{n \ln \left( 1 + \frac{r}{n} \right)}.
\]

**Example 1.14.** (Tripling an investment: discrete case) A sum of money is invested at an interest rate of 6%, compounded monthly. How many years does it take for it to triple?

**Solution:** Using the above formula, we get

\[
t = \frac{\ln 3}{n \ln \left( 1 + \frac{r}{n} \right)} = \frac{\ln 3}{12 \ln(1 + 0.005)} = 18.356 \text{ years}
\]
Continuous compounding: Start with equation (1.3).

\[ F = Pe^{rt} \quad \Rightarrow \quad \frac{F}{P} = e^{rt} \quad \Rightarrow \quad \ln\left(\frac{F}{P}\right) = \ln[e^{rt}] = rt \]

Thus,

\[ t = \frac{1}{r} \ln\left(\frac{F}{P}\right). \quad (1.11) \]

**Example 1.15.** (Increasing an investment \( m \)-fold: continuous case) We can use this formula to answer the question: How long does it take to increase an investment from \( P \) dollars to \( F = mP \) dollars at an interest rate of \( r \)?

**Solution:** Setting \( F = mP \) in the preceding formula yields

\[ t = \frac{\ln(m)}{r}. \quad (1.12) \]

**Example 1.16.** (Tripling an investment: continuous case) A sum of money is invested at an interest rate of 6%, compounded continuously. How many years does it take for it to triple?

**Solution:** Using the above formula, we get

\[ t = \frac{\ln 3}{0.06} = 18.310 \]

Compare this with the result found in example 1.14. Notice that the time is shorter in the continuous case. Does this make sense?

### 1.7.4 Time to double your money and The rule of 69 and 72

All things being equal, the faster the growth rate, the faster you can double your money.

Investors sometimes use “the rule of 72” to do a back-of-the-envelope calculation for the time it takes for your investment to double at a given interest rate. The rule of 72 says that the time it takes your money to double is roughly \( 72/r \), where \( r \) is the annual interest rate.

We have now seen that the precise value of the doubling time \( t_{\text{doubling}} \) is

\[ t_{\text{doubling}} = \frac{\ln 2}{n \ln(1 + \frac{r}{n})} \quad (\text{for the discrete case}), \]

\[ t_{\text{doubling}} = \frac{\ln 2}{r} \quad (\text{for the continuous case}), \]

where \( \ln 2 \approx 0.6931471806 \).
Question: Where does the 72 come from?
Answer: I’m not sure!

In the case of discrete compounding, the rule 72 seems to give reasonable answers over a large range of interest rates. You can test the rule of 72 using Excel. Make one column be the exact value and the second column be the approximation \( \frac{72}{r} \). You should experiment with the parameters \( n \) and \( r \) (see rule-of-72.xls).

However, for the case of continuously compounded interest it is clear that it should be the “rule of 69”! The error between the continuous case and the approximation is

\[
\frac{(72 - \ln 2)}{r} \approx \frac{(72 - .69)}{r} = .03
\]

Notice that this error is fixed and much larger than the error would be if we just used 69 instead of 72. In fact, if we compare the rule 72 to the rule of 69 over \( r \) values ranging from .01 to .20, incrementing them by a step size of \( \Delta r = .01 \) (i.e. \{.01, .02, \ldots , .19, .20\}) and over the values \( n = 1, 4, 12, 365 \) corresponding to yearly, quarterly, monthly, and daily, we find that the average absolute error in the rule of 72 is approximately .41, whereas the error in the rule of 69 is only .17 (see rule-of-72.xls). I vote that we use the rule of 69. Moreover, the rule of 69 is almost exact for the continuous case.

Rule of Thumb: When it comes to choosing when to use the rule of 72 or the rule of 69 my advice is to use the rule of 69 in the case of compound interest, and discrete compounding with \( n \geq 12 \). Use the rule of 72 for all other cases.

Advanced Comment: To take this analysis further would require using mathematical analysis. The next step would require developing an approximation based on the fact that \( r/n \) is a small number and so we can expand the exact solution in a Taylor series in the small parameter \( r/n \) about \( r/n = 0 \). For \( r/n \) small, it can be shown that

\[
\frac{\ln m}{n \ln (1 + \frac{r}{n})} \approx \frac{\ln(m)}{r} \left(1 + \frac{r}{2n} + O\left(\frac{r}{n}\right)^2\right).
\]

If \( r/n = .08 \) and \( m = 2 \) then \( \ln 2 \cdot 1.04 \approx .72 \). Since many interest rates are in the range \( r = .08n \), this might be why the rule of 72 is used so much in practice.
1.7.5 Effective Annual Yield

Because compound interest is affected by both the nominal interest rate \(r\) and the frequency of compounding \(n\), it is sometimes hard to tell offhand which of two compound interest procedures is more advantageous. For example, is it better to invest at 4\% compounded monthly or 4\% \(\frac{1}{4}\) \% compounded semiannually? It would be nice to have a common ground on which to compare various interest rates. The solution was to invent the effective annual yield, or annual percentage yield (APY), which we’ll call \(y\). It is defined as follows:

\[
P \left(1 + \frac{r}{n}\right)^n = F = P(1 + y).
\]  

(1.13)

Using the formula for simple interest \(F = P(1 + yt)\), \(y\) is the rate that would give the same future value as \(F = P(1 + r/n)^t\) for \(t = 1\) year and with principal \(P\). In words, if an investment earns compound interest, then the effective annual yield is the simple interest rate that yields the same amount at the end of one year.

Solving for \(y\) in equation (1.13) gives

\[
y = \left(1 + \frac{r}{n}\right)^n - 1.
\]  

(1.14)

The APY \(y\) is the actual percentage rate earned. It is sometimes just called the yield. Since most people understand the simple interest formula \(I = Prt\), the annual yield has the appeal that it is accessible to the common person. In fact, the Truth in Lending Law enacted in 1969 requires the annual yield \(y\) to appear on all contracts.

For continuous compounding, we can derive the effective annual yield as follows: First, equate the two expressions for \(F\), then divide by \(P\), and finally solve for the yield \(y\)

\[
Pe^r = F = P(1 + y)
\]

\[
\frac{\div P}{\downarrow} \quad e^r = 1 + y
\]

\[
\frac{\div 1}{\uparrow} \quad y = e^r - 1
\]  

(1.15)

NOTE: The effective annual yield is independent of the principal in both the discrete and continuous case.

Given a certain yield \(y\), suppose we want to find the annual rate \(r\) for the case of discrete compounding. This would require solving (1.13) for \(r\).

\[
\frac{\div P}{\downarrow} \quad \left(1 + \frac{r}{n}\right)^n = 1 + y
\]

\[
\frac{\div \left(\frac{1}{n}\right)}{\uparrow} \quad \left(1 + \frac{r}{n}\right) = (1 + y)^\frac{1}{n}
\]

\[
\frac{\div \times n}{\downarrow} \quad r = n \left[(1 + y)^\frac{1}{n} - 1\right]
\]
Example 1.17. What annual rate \( r \), compounded quarterly, has a yield of 7.5%?

Solution: Given \( n = 4 \) and \( y = 0.075 \).

\[
    r = 4[(1 + 0.075)^{1/4} - 1] = 0.073.
\]

Example 1.18. What annual rate \( r \), compounded continuously, has a yield of 7.5%?

Solution: Given continuous compounding and \( y = 0.075 \).

\[
    r = \ln(y + 1) = \ln(1.75) = 0.072.
\]

We end this section with the derivation of several useful formulas for approximating the yield. These formulas will then be applied to an example problem that is similar to a test question on a multiple-choice 20-question 40-minute exam where you only have a 4-button calculator, or no calculator at all.

Recall that in section 1.6, we gave formulas for approximating \((1 + x)^a\) and \(e^x\) by two-term polynomials related to the simple interest formula. Referring to approximation (1.6) with \( t = 1 \), we have

\[
    \left(1 + \frac{r}{n}\right)^n \approx 1 + r \quad \overset{\text{−1}}{\rightarrow} \quad y = \left(1 + \frac{r}{n}\right)^n - 1 \approx r. \quad (1.16)
\]

and referring to approximation (1.8) with \( t = 1 \) we have

\[
    e^r \approx 1 + r \quad \overset{\text{−1}}{\rightarrow} \quad y = e^r - 1 \approx r. \quad (1.17)
\]

These results show that for small values of \( r \) (say \( r < 0.2 \)), \( y \approx r \) regardless of whether we’re looking at the yield from discrete compounding or continuous compounding. This is a very powerful piece of information! We will now make immediate use of this valuable knowledge.

Example 1.19. Given the annual interest rate \( r \), determine which of the following investment opportunities will have the highest effective annual yield. Your choices are

(a) 5% compounded daily
(b) 5% compounded continuously
(c) 5% compounded quarterly
(d) 10% compounded yearly

Solution: Since all or the \( r \)-values are much less than 20%, we can apply our approximation. If we use our approximation \( y \approx r \), then in (a)-(c) \( y \approx 0.05 \), whereas in (d) \( y \approx 0.1 \). Thus, the answer is (d).

Notice that of the choices (a)-(c), (b) will have the largest yield since all of the \( r \)-values are the same and continuous compounding produces a larger growth rate than discrete compounding.
Example 1.20. Given the annual interest rate $r$, determine which of the following investment opportunities will have the lowest effective annual yield. Your choices are 

(a) 7% Compounded daily  
(b) 7% Compounded continuously

(c) 7% Compounded quarterly  
(d) 6% Compounded continuously

Solution: Since all or the $r$-values are much less than 20% we can apply our approximation. If we use our approximation $y \approx r$, then in (a)-(c) $y \approx 0.07$, whereas in (d) $y \approx 0.06$. Thus, the answer is (d). Our approximation is safely accurate to 1%.

Notice that of the choices (a)-(c), (c) will have the smallest yield, since all of the $r$-values are the same and discrete quarterly compounding grows at a slower rate than continuous and daily compounding.

How do we know when it’s O.K. to use the approximation and how do we know when it’s too close to call? The answer to this question is technical. It requires comparing the next term in the series to the difference in the annual interest rates $r$ for each answer. An easier approach to this is to argue as follows: The error as a function of $r$ in approximating the yield $y$ corresponding to continuous compound interest is $E_{\text{continuous}}(r) = y_{\text{continuous}} - r = e^r - 1 - r$. The error as a function of $r$ in approximating the yield corresponding to discrete compounding is $E_{\text{discrete}}(r) = y_{\text{discrete}} - r = (1 + r/n)^n - 1 - r$. Since the continuous compound interest grows faster than discrete compounding, it follows that the error in approximating the yield from continuous compound interest $E_{\text{continuous}}(r)$ is greater than the error in approximating the yield from discrete compounding $E_{\text{discrete}}(r)$. Thus we will plot the error function corresponding to continuous compound interest and use it to decide our cutoff value for the differences in the $r$ values that are given in the multiple choice answers. Since the error corresponding to the discrete case is smaller than that for the continuous case, this analysis will hold for the discrete case as well. We are taking an approach that it is better to err on the side of caution than to produce the wrong answer. After all, we can also find a new approach to getting a wrong answer.
Using the graph of the error, we can devise a rule of thumb for this situation. So long as the $r$ values differ by no more than 2%, and are in the range of 0 to 20%, you should be able to approximate the yield by the annual interest rate $r$. If the $r$ values in the multiple-choice answers are between 0 and .1 (10%), then the error is less than 1%. If the gap is tightened further, then we must refer to the graph of the error. But this is not very practical on an exam.

Using this information, we see that our approximation in example 1.20 is sound so long as we remember that if the $r$ values are between 0 and .01, then the error is less than 1%. For $.06 \leq r \leq .07$ the approximation has an error much less than .005, or .5%. However, in the next example the numbers are too close to call.

**Example 1.21.** Given the annual interest rate $r$, determine which of the following investment opportunities will have the lowest effective annual yield. Your choices are

(a) 7% Compounded daily 
(b) 7% Compounded annually 
(c) 7% Compounded quarterly 
(d) 6.99% Compounded continuously

We cannot use our method on this problem because the difference between .0700 and .0699 is .0001 and this is much less than .005. Even in this case we can throw out (a) and (c), since if all the $r$ values are equal, then an investment with an annually compounded interest grows slower than an investment with quarterly or daily compounded interest. The answer is either (b) or (d). But without keeping many more terms in the approximation, or having the use of a scientific calculator, we cannot be sure which one is the correct answer. But do not despair! The good news is that the test makers cannot give you a question like this if you are only armed with a 4-button calculator.
1.8 Present value and the risk-free rate of return

[ADD: define risk-free rate of return]

Sometimes we want to know the amount \( P \) that we should invest today in order for our investment to be worth \( F \) at a particular time in the future. For this reason \( P \) is sometimes called the present value and \( F \) is called the future value. For example, suppose a Treasury bill will be worth 1000 dollars one year from now. The interest rate on the T-bill is 5% compounded continuously. What is a fair price for this T-bill? We’ve already seen how to answer such a question.

**Example 1.22.** Suppose your house burned down and you wanted to buy another house. You found a house that you’d like to buy, but the sellers will not be able to move out for 6 months. They have another potential buyer, so they demand that you pay for the house today. You estimate that the house will be worth $200,000 (the price of houses in a new neighborhood nearby). If the interest rates on Treasury bills are 5% compounded continuously, what is a fair price for the house?

**Solution:**

\[
P = Fe^{-rt} = 200000e^{-0.05 \cdot 0.5} = 200000e^{-0.025} = \$195,062.
\]

When you buy a US Treasury bill that matures in a few months, you’re making a risk-free investment. The amount you earn on this investment is the risk-free return. The interest rate that the T-Bill offers is known as the risk-free rate of return. By subtracting the risk-free return from the return on an investment that has the potential to lose value, you can figure out the risk premium, which is one measure of the risk of choosing an investment other than the short term T-bill.

Since the value of any T-bill is backed by the US Government (which also guarantees the worth of the dollar), it is considered a risk-free investment. The only practical danger in buying a long-term T-bill is that the guaranteed rate of return may not keep up with inflation. Over short terms like 6 months, the typical life of an option, inflation isn’t likely to change that much; so a T-bill is a good measure of what your money would be worth 6 months from now if you didn’t want to take any risks. The key word here is risk. You cannot make a large return on your money without incurring some level of risk. The more risk in the investment, the more money (paid in terms of interest) that you will demand. Remember, companies that want you to invest in their stocks must make a bid for your money, and they do that by paying higher interest rates. If I can make a safe investment in a fixed-index fund like the S&P 500 and earn roughly an 8% return, why would I invest in your high-risk stock if you were only offering a return of 8% as well? The answer is: I wouldn’t! You would have to offer me a much higher interest rate to entice me to buy your stock.

Many people think they make money in the stock market because they have a knack for picking good stocks. Except in rare instances, I disagree with this philosophy. The reason people make money in the stock market is because the stock market must on
average pay a higher rate of return than safer investments like T-bills, bonds, and banks. If the average rate of return on an investment was low, then only a very few people (the type that like to gamble) would invest in the stock market. Most investors would leave their money in the bank.

1.9 Variable interest rates

The following subsection is advanced material. This material may be skipped without jeopardizing your understanding of future material.

So far, we’ve assumed that the simple interest rate $r$ remains constant over the life of an investment.

That’s a valid assumption for many forms of investment. For example, most CD’s and bonds pay interest at a fixed rate.

On the other hand, many investments do not. There are CD’s whose rates change to reflect inflation. Adjustable-rate mortgages vary their rates, generally in step with the prime lending rate, the LIBOR, or some other measure of the current market rate. Money-market accounts pay at a different rate almost every month. We would like to know how to calculate and compare the yields of such investments.

1.9.1 Variable interest rates: discrete compounding

Let’s begin with a fairly simple example. We’ll assume you’ve paid $10,000 for a six-month CD paying simple interest at 5%. At the end of the six months, you let the CD renew automatically. When this happens, the bank can change the rate: in our case, the rate goes up to 6% for the second six months. What will the value of the CD be at the end of the year?

We’ll look at this in steps. At the end of the first six months, the value of the CD is

$$F = P(1 + rt) = 10,000 \left( 1 + \frac{0.05}{2} \right) = 10,250$$

For the second six-month period, the principal $P$ is $10,250. Applying the simple-interest formula again, we get

$$F = P(1 + rt) = 10,250 \left( 1 + \frac{0.06}{2} \right)$$
$$= 10,000 \left( 1 + \frac{0.05}{2} \right) \left( 1 + \frac{0.06}{2} \right) = 10,557.50 \quad (1.18)$$
We can generalize this. Suppose we’ve got an investment that compounds $n$ times per year, with a different simple-interest rate in each compounding interval: in the $i$th interval, the rate is $r_i$. We start with a principal of $P$. At the end of the first interval, the simple-interest formula gives us

$$F_1 = P \left(1 + \frac{r_1}{n}\right)$$

For the second compounding interval, the principal is $F_1$, the value of the account at the beginning of the interval. The simple-interest formula gives us

$$F_2 = F_1 \left(1 + \frac{r_2}{n}\right) = P \left(1 + \frac{r_1}{n}\right) \left(1 + \frac{r_2}{n}\right)$$

For the third interval, the principal is $F_2$, and we get

$$F_3 = F_2 \left(1 + \frac{r_3}{n}\right) = P \left(1 + \frac{r_1}{n}\right) \left(1 + \frac{r_2}{n}\right) \left(1 + \frac{r_3}{n}\right)$$

We can keep doing this. For the $m$th interval, the principal is $F_{m-1}$, and the simple-interest formula gives us

$$F_m = F_{m-1} \left(1 + \frac{r_m}{n}\right) = P \left(1 + \frac{r_1}{n}\right) \left(1 + \frac{r_2}{n}\right) \cdots \left(1 + \frac{r_m}{n}\right)$$ \hspace{1cm} (1.19)

Is there an easier way to do this? For example, couldn’t we just use the average of the different interest rates? Let’s look back at our first example, on page 32. There, we had two six-month periods. The simple-interest rate during the first period was 5%, and during the second period it was 6%. The average of those two is 5.5%; suppose we just used 5.5% for the whole year?

First of all, we can’t use 5.5% simple interest for the whole year. That would give us a year-end value of

$$F = 10,000 \left(1 + 0.055\right) = 10,550$$

The actual value we found in Equation (1.18) was $10,557.50; so the easy way doesn’t work.

Suppose we try a slightly more complicated trick? Let’s use that average of 5.5% and compound it semi-annually. Then we’d get

$$F = 10,000 \left(1 + \frac{0.055}{2}\right)^2 = 10,557.56$$

This is close, but it’s still wrong. The only correct approach to this situation is Equation (1.19).
Example 1.23. You invest $43,000 in a money-market account that compounds monthly. For the first month, the average yield is 3.72%; for the second, it’s 3.91%; for the third, it’s 4.00%; and for the fourth month, it’s 4.19%. What is the value of your account at the end of the fourth month?

Solution: Use Equation (1.19). Since you’re compounding monthly, \( n = 12 \).

\[
F = 43,000 \left(1 + \frac{0.0372}{12}\right) \left(1 + \frac{0.0391}{12}\right) \left(1 + \frac{0.0400}{12}\right) \left(1 + \frac{0.0419}{12}\right)
\]

\[
= 43,569.69
\]

1.9.2 Variable interest rates: continuous compounding

Now, let’s modify our first example. You’re again buying a $10,000 CD at 5% for six months, then renewing it at 6% for another six months; but this time, the interest is continuously compounded.

We’ll use the continuous-compounding formula, Equation (1.3), to get the value of the account after the first six months:

\[
F = Pe^{rt} = 10,000 e^{0.05/2} = 10,253.15
\]

Again, for the second six months, the principal is the value at the end of the first six months; and the value of the account at the end of the year is

\[
F = 10,253.15 e^{0.06/2} = 10,000 e^{0.05/2} e^{0.06/2} = 10,565.41
\]

Remember the property of exponents:

\[
e^a e^b = e^{a+b}
\]

That lets us write

\[
F = 10,000 \exp \left( \frac{0.05 + 0.06}{2} \right)
\]

(The notation “exp” allows us to write exponentials without tiny hard-to-read exponents. We can write \( \exp(x) \) instead of \( e^x \) to make our equation easier to read.)

Notice that in this case, we can use the average of the interest rates. Of course, that depends on the two periods being of equal length. There’s no reason why that has to be so.

Suppose we have continuously compounding interest, with a rate that changes with time. For the first \( t_1 \) years, the rate is \( r_1 \); for the next \( t_2 \) years, it’s \( r_2 \); and so on up to the last \( t_m \) years, for which it’s \( r_m \). The initial principal is \( P \).
At the end of the first $t_1$ years, Equation (1.3) gives us the value of the account:

$$F_1 = Pe^{r_1t_1} = P \exp(r_1t_1)$$

As with discrete compounding, the principal for the next $t_2$ years is $F_1$, and the value at the end of $t_1 + t_2$ years is

$$F_2 = F_1 e^{r_2t_2} = Pe^{r_1t_1} e^{r_2t_2} = P \exp(r_1t_1 + r_2t_2)$$

We can keep going like this until we get

$$F_m = F_{m-1} e^{r_mt_m} = Pe^{r_1t_1} e^{r_2t_2} \cdots e^{r_mt_m} = P \exp(r_1t_1 + r_2t_2 + \cdots + r_mt_m) = P \exp \left( \sum_{i=1}^{m} r_it_i \right),$$

where the total time interval over which the investment grows is $t_1 + \cdots + t_m$ years.

Notice that all of these variable interest rate formulas, although messy, are defined recursively. It is for this reason that they can easily be coded into Excel, or some other programming language. By using a standard nested loop, you could input the time intervals $t_i$ and the interest rates over the time intervals to arrive at a recursive (repeating) algorithm to compute the $F_i$’s.

Let’s take this analysis a little further. To have a complete understanding of what I am about to present, the reader will need to have a solid understanding of calculus. However, even those without calculus in their backgrounds should be able to glean some useful information from the following discussion.

### 1.9.3 Continuous Rate Change-Continuous Compounding

Suppose we want to compute the earnings of an investment of $P$ dollars over $T$ years. Furthermore suppose that the interest scheme is compounded continuously and that the interest rate changes over very small time scales that need not be of equal length. This case is more subtle than the previous cases, but it is not even close to how much more complicated things can get. For example, the analysis can get much worse if we treat the interest rate as a random variable. To solve this case we will need to take the limit as the time intervals shrink to zero. This case is addressed in calculus. It requires the use of the Riemann integral.

We begin by partitioning the time interval $[0, T]$ (time is given in years) into $n$ subintervals $[t_{i-1}, t_i]$ of length $\Delta t_i = t_i - t_{i-1}$, where $i = 1, 2, \ldots, n$, the length of the interval is denoted by $\Delta t_i$, and $t_0 = 0$ and $t_n = T$. That is, we want to cut up the original interval into little disjoint pieces.
We assume that we partition the interval into so many small partitions that the length of the largest partition, denoted by $\Delta t_{\text{max}}$, is made arbitrarily small. For example, if we take the interval $[0,1]$ (one year) and divide the interval into $n$ subintervals of equal length, then the width of each subinterval is $\Delta t = \frac{1-0}{n} = \frac{1}{n}$. For larger and larger values of $n$ (more and more partitions of the original interval), the expression $\frac{1}{n}$ goes to zero and so the width of the subintervals goes to zero. In this case there was not a unique largest subinterval, since all of the subintervals were of equal length; however, it should be clear that if we subdivide the original interval in such a way that the maximum subinterval gets smaller and smaller, then all of the subintervals are getting smaller and smaller.

Notice that by the construction process of our subintervals, that even though the width of each subinterval gets smaller and smaller, the sum of the widths of all of the subintervals taken as a whole is the length of the total interval $T - 0$.

We will now outline the details of such a construction. Let $t^*_i$ be any time value with the property that $t_{i-1} < t^*_i < t_i$. Then as we increase the number of partitions, $t^*_i$ approaches $t_i$, and so $r(t^*_i)$ is a close approximation to the function $r(t)$ over this interval. The area under the curve $y = r(t)$ over the interval $t_{i-1} < t^*_i < t_i$ can be approximated by the rectangle with base $\Delta t_i$ and height $r(t^*_i)$. The future value is given by

$$F_n = F_{n-1} e^{r(t^*_i)\Delta t_i} = P e^{r(t^*_1)\Delta t_1} e^{r(t^*_2)\Delta t_2} \cdots e^{r(t^*_n)\Delta t_n} = P \exp\left(\sum_{i=1}^{n} r(t^*_i) \Delta t_i\right).$$

Taking the limit as the number of partitions become finer and finer (i.e. $n \to \infty$ and $\Delta t_{\text{max}} \to 0$) will yield the Riemann integral:

$$\lim_{n \to \infty} \sum_{i=1}^{n} r(t^*_i) \Delta t_i = \int_{0}^{T} r(t) \, dt.$$

It is important to remember that this is a process where to each new value of $n$ we repartition the interval $[0,T]$. As the process continues the Riemann sums approach the Riemann integral. The above expression for the future value then becomes:

$$F(T) = P \exp\left(\int_{0}^{T} r(t) \, dt\right).$$

By the mean-value theorem, there exists a time $t_*$:

$$r(t_*) = \frac{1}{T} \int_{0}^{T} r(t) \, dt.$$
If we define $\bar{r} = r(t_*)$, then we may rewrite the equation above equation as

$$F(T) = Pe^{\bar{r}T},$$

(1.21)

where $\bar{r} = \frac{1}{T} \int_0^T r(t) \, dt$ is the value of the interest rate averaged over the interval. Notice that this expression formally looks like the classic "Pert" formula for continuous compound interest: $F = Pe^{rt}$.

Through a limiting process we have derived a formula for the case of continuous compounding with a variable interest rate $r(t)$ that changes continuously. It has many similarities with the discrete case under continuous compounding, and this is no accident since it was derived as the limit of the discrete case.

We now summarize what we have done. If we define the annual interest rate $\bar{r}$ to be the average value of the continuously changing interest rate $r(t)$ over the interval $[0, t]$ for any $t \in \mathbb{R}$, then

$$\bar{r} = \frac{1}{t} \int_0^t r(s) \, ds$$

then substituting for $\bar{r}$ in equation (1.3) yields

$$F = Pe^{\bar{r}t} = Pe^\left(\frac{1}{t} \int_0^t r(s) \, ds\right) = Pe^{\int_0^t r(s) \, ds}.$$ 

We can understand this formula as the limit of the discrete case. It is the end result of a process of generating a Riemann integral from a Riemann sum in the limit as $n \to \infty$ together with a proper repartitioning of the interval $[0, t]$.

For such a situation we would probably not want to let $t$ get too large, say $t = 1$ year, since a long-time average defined this way would probably not hold much practical meaning. It would be better to use a moving average to examine long term trends.

Fortunately for you, this course does not concern itself with variable interest rates. From here forward, we will assume a constant interest rate for all time. This assumption is reasonable for Bonds, Bills and cds, but not for stocks and money market accounts.
1.10 Variable interest rate over each compounding interval

We begin by partitioning the time interval \([0, T]\) (time is given in years) into \(n\) subintervals \([t_{i-1}, t_i]\) of equal length, where \(i = 1, 2, \ldots, n\). The width of each subinterval, denoted by \(\Delta t\), is the length of the total interval \(T - 0\) divided by the number of times that we want to partition it \(n\). Thus \(\Delta t_i = T/n\). Then \(t_i = t_{i-1} + \Delta t\). Using \(t_1 = T/n\) and recursion, we find \(t_i = \frac{i}{n}T\) for \(i = 1, \ldots, n\). As a check, notice that \(\Delta t = t_i - t_{i-1} = \frac{T}{n}\).

Next we compute the earnings ratio over each subinterval \([t_{i-1}, t_i]\). Since the rate is constant over each subinterval, this will lead to the following results:

The future value over the \(i^{th}\) period is given by

\[
F(t_i) = F(t_{i-1})(1 + r_i \Delta t) = F(t_{i-1})(1 + Tr_i/n)
\]

if discrete compounding is used over the interval, and

\[
F(t_i) = F(t_{i-1})e^{r_i \Delta t} = F(t_{i-1})e^{Tr_i/n}
\]

if continuous compounding is used over the interval. The earnings ratios become:

\[
\frac{F(t_i)}{F(t_{i-1})} = (1 + Tr_i/n) \quad \text{(discrete compounding)}
\]

\[
\frac{F(t_i)}{F(t_{i-1})} = e^{Tr_i/n} \quad \text{(discrete compounding)}
\]

**Discrete Rate Change-Discrete Compounding:**

\[
F = P \left( 1 + T\frac{r_1}{n} \right) \left( 1 + T\frac{r_2}{n} \right) \ldots \left( 1 + T\frac{r_n}{n} \right) \quad (n \text{ factors}) . \tag{1.22}
\]

If

\[ r = r_1 = r_2 = \cdots = r_n \]

then the interest rate is constant and for \(T = 1\) formula (1.2) is valid.

As we’ve already seen in the example above, to define the nominal interest rate \(r\) in terms of the arithmetic-average

\[ r = \frac{\sum_{i=1}^{n} r_i}{n} = \frac{\sum_{i=1}^{n} r_i}{n} \]

is not very useful in practice, and in general leads to an over estimate of the effective annual interest rate. The reason for this over estimate will be given shortly. Perhaps a better approach is the set the expression in 1.22 equal to an equivalent expression with
constant interest

**Discrete Rate Change-Continuous Compounding:** Suppose that over each of the \( n \) intervals that the interest is compounded continuously. The earnings ratio over the \( i^{th} \) period is \( F/P = e^{r_i \Delta t_i} = e^{r_i T/n} \), since the duration of this time interval is \( \Delta t_i = T/n \). Then the earnings ratio at the end of the \( n \) periods becomes:

\[
\frac{F}{P} = e^{\frac{r_1}{n}} e^{\frac{r_2}{n}} \cdots e^{\frac{r_n}{n}} = e^{r_1 T/n + r_2 T/n + \cdots + r_n T/n} = \exp \left( T \sum_{i=1}^{n} \frac{r_i}{n} \right).
\]

If we define the *annual arithmetic-average interest rate* as

\[
r = \frac{\sum_{i=1}^{n} r_i}{n} = \frac{\sum_{i=1}^{n} r_i}{n}.
\]

Then the resulting mathematically equivalent expression superficially resembles the standard compound interest formula

\[
\frac{F}{P} = e^{r T}.
\]

### 1.10.1 Geometric mean

Let’s start with a problem:

**Problem:** John and Mary each invested $10,000 for two years, compounded annually. John’s investment paid interest of 10% for the first year, and no interest for the second. Mary’s investment paid 5% in the first year, and 5% in the second. Which one made the better investment?

If you weren’t taking Business Math, your answer would probably be something like, “The average yield of both investments is 5%, so neither investment was better than the other.” However, as a Business Math student, you distrust this easy answer.

You’re right to be distrustful. Let’s look at the changes in both investments over the two years.

John’s investment yields 10% over the first year. At the end of the first year, the value is

\[ F = 10,000(1 + 0.10) = 11,000 \]

In the second year, his investment pays no interest at all; so the value at the end of the second year is still $11,000.
Meanwhile, Mary’s investment yields 5% in the first year; so at the end of the first year, it’s worth

\[ F = \$10,000(1 + 0.05) = \$10,500 \]

In the second year, it also yields 5%, so at the end of two years its value is

\[ F = \$10,500(1 + 0.05) = \$11,025 \]

Obviously, Mary’s investment was better than John’s, since she’s got $25 more than he does at the end of the two years.

It would be a good thing if we had some kind of “average” yield of investments like John’s, so that we could compare them with Mary’s 5% average yield. Fortunately, we have such a thing. We’re looking for a constant rate \( r \) that, over the two-year life of the investment, produces the same final value as the 10% yield in the first year and 0% in the second. In other words:

\[
(1 + r)^2 = (1 + 0.10)(1 + 0.00) = 1.10 \\
1 + r = (1.10)^{1/2} = 1.0488 \\
r = 4.88\% 
\]

Thus John’s investment was equivalent to one that yielded 4.88% for both years. Since Mary’s investment yielded 5% for both years, hers was the better one.

An “average” like we’ve just described is called the geometric mean. We can extend the definition of it for an arbitrary number of years. Suppose we hold a certain investment for \( n \) years. In the first year, it produces a return of \( r_1 \); in the second year, of \( r_2 \), \ldots; in the \( n \)th year, it returns at a rate of \( r_n \). Then after \( n \) years, its value is

\[ F = P(1 + r_1)(1 + r_2)\cdots(1 + r_n) \]

We want to find a constant rate \( r \) that will produce the same future value over \( n \) years:

\[
(1 + r)^n = (1 + r_1)(1 + r_2)\cdots(1 + r_n) \\
1 + r = \left[ (1 + r_1)(1 + r_2)\cdots(1 + r_n) \right]^{1/n} \\
r = \left[ (1 + r_1)(1 + r_2)\cdots(1 + r_n) \right]^{1/n} - 1 \quad (1.23) 
\]

This \( r \) is the geometric mean of \( r_1, r_2, \ldots, r_n \).

The “average” that you’re used to is called the arithmetic mean. For \( r_1, r_2, \ldots, r_n \), it’s

\[ r_A = \frac{r_1 + r_2 + \cdots + r_n}{n} \]

Using calculus, it can be shown that the geometric mean is always less than or equal to the arithmetic mean. In fact, careless use of the arithmetic mean can get you in serious trouble.
Example 1.24. John and Mary both invest $10,000 for two years. John chooses a high-risk investment that yields 90% in the first year, then loses 50% in the second year. Mary’s investment yields 5% in both years. Which one made the better investment?

We haven’t looked at a loss of money before, but it’s not complicated. You still use the simple-interest formula
\[ F = P(1 + rt) \]
but when you lose money, \( r \) is negative.

If you were naive (which you’re not), you’d say: “John’s investment yields an average of
\[ r_A = \frac{0.90 - 0.50}{2} = 0.20 = 20\% \]
and Mary’s yields an average of 5% per year; so John’s must be a lot better.”

As a Business Math student, you know better. Over the two years, the value of Mary’s investment becomes
\[ F = 10,000(1 + 0.05)(1 + 0.05) = 11,025 \]
In the same time, the value of John’s investment becomes
\[ F = 10,000(1 + 0.90)(1 - 0.50) = 9746.79 \]
Not only has John’s investment not performed as well as Mary’s; he’s actually lost money. Poor John! If only he’d known about the geometric mean:
\[
(1 + r)^2 = (1 + 0.90)(1 - 0.50) = 0.974679 \\
1 + r = (0.974679)^{1/2} = 0.98726 \\
r = -0.0127 = -1.27\% 
\]
In other words, John’s investment is equivalent to one that loses 1.27% in each of the two years.

Example 1.25. The value of an investment grows by 5% in the first year, by 10% in the second year, and by 15% in the third year. What is the equivalent constant rate \( r \) that would produce the same yield over the three years?

Solution: We are looking for the geometric mean here. Equation (1.23) gives us:
\[
(1 + r)^3 = (1 + 0.05)(1 + 0.10)(1 + 0.15) = 1.32825 \\
1 + r = (1.32825)^{1/3} = 1.0992 \\
r = 0.992 = 9.92\% 
\]
Of course, you’ve noticed that this is less than the arithmetic mean of the three rates, which is 10%.
1.10.2 Approximating the geometric mean by the arithmetic mean

Under the right circumstances the geometric mean can be approximated by the arithmetic mean. This follows directly from a multi-dimensional Taylor series expansion of the geometric mean. In particular, if $r_1, \ldots, r_n$ are all much, much less than one, denoted mathematically as $r_i \ll 1$, for $i = 1, \ldots, n$, then

$$(1 + r_1)(1 + r_2) \cdots (1 + r_n) \approx 1 + \sum_{i=1}^{n} r_i + \text{error terms involving the product of the } r'_i\text{.}$$

Using the Binomial expansion formula it can be shown that

$$r = \left[ (1 + r_1)(1 + r_2) \cdots (1 + r_n) \right]^{1/n} - 1 \approx 1 + \frac{1}{n} \sum_{i=1}^{n} r_i - 1 = \frac{1}{n} \sum_{i=1}^{n} r_i.$$ 

Thus, in the case of small $r_i$ it can be shown that the geometric mean can be approximated by the arithmetic mean.

1.11 A last word on interest rates

Interest may be defined as the compensation that a borrower of capital pays a lender of capital. It is sometimes expressed as a percentage of the sum borrowed. The interest rate $r$ is sometimes referred to as the nominal interest rate, or annual interest rate. By nominal, we mean the amount or face value of a sum of money or a bond, for example, and not the purchasing power or market value. These are just decorative adjectives for the rate of interest on an investment without any adjustment for compounding or inflation. These interest rates are measured in dollars, not in material goods like food and shelter. Nominal interest is the return on an investment measured in dollars per year per dollar of investment. But dollars can become distorted units of measure under inflation. They are not absolutes like food and shelter. An interest rate that is adjusted for inflation is known as a real interest rate. Thus a real interest rate is the nominal interest rate minus the rate of inflation.

**Advanced Comment:** The following argument requires some tools from calculus. We are going to learn about many different kinds of interest: simple, discrete compounding, and continuous compounding. Associated with these are various interest rates. But the nominal interest rate $r$ is the most fundamental. All other interest rates that we will look at are defined in terms of this interest rate. This rate is intimately linked to the growth rate of the investment. In particular, if $F(t + \Delta t)$ is the future value of the investment at time $t + \Delta t$ and $F(t)$ is the future value of the investment at time $t$, then the difference $F(t + \Delta t) - F(t)$ is the price appreciation over time $\Delta t$, where $\Delta t$ is assumed to be a very small unit of time. The rate of return on the investment over the
time period \([t, t + \Delta t]\) is defined as \((F(t + \Delta t) - F(t))/F(t)\). The nominal rate of interest can be determined from

\[
\lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t F(t)} = \frac{F'(t)}{F(t)}.
\]

This limit is independent of time for the case of discrete compound and continuous compound interest, and decreases as a function of time in the case of simple interest. Thus \(r\) is the relative rate of increase of an investment. Notice that \(F'/F\) has the dimension of one over time, and that it represents the instantaneous rate of change of the investment per unit dollar of investment. That is, the units are dollars per year per dollar of investment, which is just one over time (measured in years), hence the name annual interest rate.
1.12 Compound Interest: Practice Problems

Round all answers to the nearest whole dollar, or to the nearest 0.01%.

Exercise 1 (Compound Interest): 
(i) $56,000 is invested at 3.56%, compounded quarterly. What is the account’s value after 5 years and 3 months?  
(ii) What is the effective annual yield on the investment?

Exercise 2 (Compound Interest): 
(i) $110,000 is invested at 5.25%, compounded monthly. What is the value of the account after 4 years?  
(ii) What is the effective annual yield on the investment?

Exercise 3 (Compound Interest): 
(i) $74,000 is invested at 4.8%, compounded daily. What is the value of the account after 6 years and 6 months?  
(ii) What is the effective annual yield on the investment?

Exercise 4 (Compound Interest): A certificate of deposit pays interest at 3.60%, compounded quarterly. What is the effective annual yield?

Exercise 5 (Compound Interest): A certificate of deposit pays interest at 4.25%, compounded monthly. What is the effective annual yield?

Exercise 6 (Compound Interest): A certificate of deposit pays interest at 5.10%, compounded daily. What is the effective annual yield?

Exercise 7 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded quarterly. If the effective annual yield is 3.35%, what is the value of $r$?

Exercise 8 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded monthly. If the effective annual yield is 5.80%, what is the value of $r$?

Exercise 9 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded daily. If the effective annual yield is 3.58%, what is the value of $r$?

Exercise 10 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded annually. If the effective annual yield is 4.15%, what is the value of $r$?

Exercise 11 (Compound Interest): 
(i) $35,000 is invested at 3.4%, compounded continuously. What is the value of the account after 7 years?  
(ii) What is the effective annual yield on the investment?
Exercise 12 (Compound Interest): (i) $120,000 is invested at 5.02%, compounded continuously. What is the value of the account after 2 years and six months? (ii) What is the effective annual yield on the investment?

Exercise 13 (Compound Interest): Two certificates of deposit have the same effective annual yield. The first pays a rate of \( r \), compounded monthly. The second pays 6.03%, compounded continuously. What is the value of \( r \)?

Exercise 14 (Compound Interest): Two certificates of deposit have the same effective annual yield. The first pays a rate of 4%, compounded quarterly. The second pays a rate of \( r \), compounded continuously. What is the value of \( r \)?

Exercise 15 (Compound Interest): $5000 is invested in an account paying 5.5%, compounded quarterly. How long will it take for the value of the account to reach $6000? Round your answer to the nearest hundredth of a year.

Exercise 16 (Compound Interest): $12,000 is invested in an account paying 3.85%, compounded monthly. How long will it take for the value of the account to reach $16,000? Round your answer to the nearest hundredth of a year.

Exercise 17 (Compound Interest): $3000 is invested in an account paying 3.2%, compounded daily. How long will it take for the value of the account to reach $5000? Round your answer to the nearest hundredth of a year.

Exercise 18 (Compound Interest): $3000 is invested in an account paying 3.2%, compounded continuously. How long will it take for the value of the account to reach $5000? Round your answer to the nearest hundredth of a year.

Exercise 19 (Compound Interest): A deposit account pays interest of \( r \), compounded continuously. If the effective annual yield is 3.91%, what is \( r \)?

Exercise 20 (Compound Interest): A deposit account pays interest of \( r \), compounded continuously. If the effective annual yield is 7.12%, what is \( r \)?

Exercise 21 (Compound Interest): Suppose money earns at an annual rate of 3.65%, compounded quarterly. What is the present value of a $30,000 payment 4 years from now?

Exercise 22 (Compound Interest): Suppose money earns at an annual rate of 5%, compounded monthly. What is the present value of a $5000 payment 6 years from now?

Exercise 23 (Compound Interest): Suppose money earns at an annual rate of 4.5%, compounded daily. What is the present value of a $125,000 payment 3 years and 6 months from now?
Exercise 24 (Compound Interest): Suppose money earns at an annual rate of 5.2%, compounded continuously. What is the present value of a $23,000 payment 5 years from now?

Exercise 25 (Compound Interest): Suppose money earns at an annual rate of 7.15%, compounded monthly. What is the future value of $13,000 after 3 years?

Exercise 26 (Compound Interest): Suppose money earns at an annual rate of 4.6%, compounded continuously. What is the future value of $2000 after 7 years and 6 months?

Exercise 27 (Compound Interest): What is the yearly ratio of future value to present value that corresponds to an annual rate of 7.6%, compounded continuously?

Exercise 28 (Compound Interest): What is the annual rate of interest \( r \), compounded continuously, that corresponds to a yearly ratio of future value to present value of 1.10?

Exercise 29 (Compound Interest): A mortgage of $120,000 is to be paid off in 360 equal monthly payments, beginning one month after the purchase date. The interest on the mortgage is 5.78%, compounded continuously.

(i) What is the amount of each payment?

(ii) What is the total amount of all the payments?

Exercise 30 (Compound Interest): A sofa costing $899.99 can be paid for in 24 equal monthly payments, beginning one month after the purchase date. The interest on the unpaid balance is 21.5%, compounded continuously.

(i) What is the amount of each payment?

(ii) What is the total amount of all the payments?
1.13 Compound Interest: Solutions to Practice Problems

NOTE: In the problems involving compound interest, $F(t) = Pe^{rt}$ (continuous), $F(t) = P(1 + \frac{r}{n})^{nt}$ (discrete). To find $F$, we need $P$, $r$, $n$, and $t$. Yield is given by $y = e^r - 1$ (continuous), $y = (1 + \frac{r}{n})^n - 1$ (discrete).

Exercise 1 (Compound Interest): (i) $56,000 is invested at 3.56%, compounded quarterly. What is the account’s value after 5 years and 3 months?

Solution. Given: $P = 56,000$, $t = 5.25$, $r = .0356$, and $n = 4$ (since compounded quarterly). Want

(i) $F(5.25) = 56,000 \left(1 + \frac{.0356}{4}\right)^{4(5.25)}$

(ii) $y = \left(1 + \frac{.0356}{4}\right)^4 - 1$

Exercise 2 (Compound Interest): (i) $110,000 is invested at 5.25%, compounded monthly. What is the value of the account after 4 years?

(i) $F(4) = 110,000 \left(1 + \frac{.0525}{12}\right)^{12(4)}$

(ii) $y = \left(1 + \frac{.0525}{12}\right)^{12} - 1$

Exercise 3 (Compound Interest): (i) $74,000 is invested at 4.8%, compounded daily. What is the value of the account after 6 years and 6 months?

(ii) What is the effective annual yield on the investment?

Solution. Given: $P = 74,000$, $t = 6.5$, $r = .048$, and $n = 365$ (since compounded daily). Want

(i) $F(6.5) = 74,000 \left(1 + \frac{.048}{365}\right)^{365(6.5)}$

(ii) $y = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{.048}{365}\right)^{365} - 1$

Exercise 4 (Compound Interest): A certificate of deposit pays interest at 3.60%, compounded quarterly. What is the effective annual yield?

Solution. Given: $n = 4$, $r = .036$. Want $y = ?$

$y = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{.036}{4}\right)^4 - 1$
Exercise 5 (Compound Interest): A certificate of deposit pays interest at 4.25%, compounded monthly. What is the effective annual yield?
Solution. Given: $n = 12$, $r = .0425$. Want $y =$?

$$y = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{.0425}{12}\right)^{12} - 1$$

Exercise 6 (Compound Interest): A certificate of deposit pays interest at 5.10%, compounded daily. What is the effective annual yield?
Solution. Given: $n = 365$, $r = .051$. Want $y =$?

$$y = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{.051}{365}\right)^{365} - 1$$

Exercise 7 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded quarterly. If the effective annual yield is 3.35%, what is the value of $r$?
Solution. Given: $n = 4$, $y = .0335$. Want $r$.

$$r = n \left[(1 + y)^{1/n} - 1\right] = 4 \left[(1 + .0335)^{1/4} - 1\right]$$

Exercise 8 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded monthly. If the effective annual yield is 5.80%, what is the value of $r$?
Solution. Given: $n = 12$, $y = .058$. Want $r$.

$$r = 12 \left[(1 + .058)^{1/12} - 1\right]$$

Exercise 9 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded daily. If the effective annual yield is 3.58%, what is the value of $r$?
Solution. Given: $n = 365$, $y = .0358$. Want $r$.

$$r = 365 \left[(1 + .0358)^{1/365} - 1\right]$$

Exercise 10 (Compound Interest): A certificate of deposit pays interest at rate $r$, compounded annually. If the effective annual yield is 4.15%, what is the value of $r$?
Solution. Given: $n = 1$, $y = .0415$. Want $r$.

$$r = 1 \left[(1 + .0415)^{1/1} - 1\right] = [(1 + .0415) - 1] = .0415 = y$$

Notice that when $n = 1$, it’s always true that $r = y$.

Exercise 11 (Compound Interest): (i) $35,000 is invested at 3.4%, compounded continuously. What is the value of the account after 7 years? (ii) What is the effective annual yield on the investment?
Solution. Given: $P = 35,000$, $t = 7$ years, $r = .034$.

(i) $F = 35,000e^{(.034)(7)}$
(ii) \[ y = e^r - 1 = e^{0.034} - 1 \]

Exercise 12 (Compound Interest): (i) $120,000 is invested at 5.02\%, compounded continuously. What is the value of the account after 2 years and six months? 
(ii) What is the effective annual yield on the investment?

Solution. Given: \( P = 120,000 \), \( t = 2.5 \) years, \( r = 0.0502 \).

(i) \[ F = 120,000e^{(0.0502)(2.5)} \]

(ii) \[ y = e^r - 1 = e^{0.0502} - 1 \]

Exercise 13 (Compound Interest): Two certificates of deposit have the same effective annual yield. The first pays a rate of \( r \), compounded monthly. The second pays 6.03\%, compounded continuously. What is the value of \( r \)?

Solution. Want \( r \) so that

\[ P \left( 1 + \frac{r}{n} \right)^n = P(1 + y) = Pe^{0.0603} . \]

Cancelling \( P \) from both sides gives \( (1 + \frac{r}{n})^n = e^{0.0603} \). Since we compound monthly, \( n = 12 \). The equation becomes \( (1 + \frac{r}{12})^{12} = e^{0.0603} \). Taking the 12th root gives

\[ 1 + \frac{r}{12} = (e^{0.0603})^{1/12} = e^{0.0603/12} . \]

Solving for \( r \) yields

\[ r = 12 \left[ e^{0.0603/12} - 1 \right] \]

Exercise 14 (Compound Interest): Two certificates of deposit have the same effective annual yield. The first pays a rate of 4\%, compounded quarterly. The second pays a rate of \( r \), compounded continuously. What is the value of \( r \)?

Solution. Want \( r \) so that

\[ Pe^r = P(1 + y) = P \left( 1 + \frac{0.04}{4} \right)^4 . \]

Cancelling \( P \) from both sides gives \( e^r = (1 + \frac{0.04}{4})^4 \). Since we compound quarterly, \( n = 4 \). The equation becomes \( e^r = (1 + \frac{0.04}{4})^4 = (1.01)^4 \). Taking the natural log of both sides gives

\[ r = \ln(e^r) = \ln((1.01)^4) = 4 \ln(1.01) \]

Exercise 15 (Compound Interest): $5000 is invested in an account paying 5.5\%, compounded quarterly. How long will it take for the value of the account to reach $6000? Round your answer to the nearest hundredth of a year.
Solution. Given: $P = 5000$, $F = 6000$, $r = .055$, and $n = 4$. Solve for $t$.

\[ F = P \left(1 + \frac{r}{n}\right)^{nt} \quad \xrightarrow{\div P} \quad \left(1 + \frac{r}{n}\right)^{nt} = \frac{F}{P} \]

\[ \xrightarrow{\ln(\cdot)} \ln \left(1 + \frac{r}{n}\right)^{nt} = \ln \left(\frac{F}{P}\right) \quad \text{(take the log of both sides)} \]

\[ \Rightarrow \quad nt \ln \left(1 + \frac{r}{n}\right) = \ln \left(\frac{F}{P}\right) \quad \text{(property of logarithms)} \]

\[ \Rightarrow \quad t = \frac{\ln(F/P)}{n \ln \left(1 + \frac{r}{n}\right)} \quad \text{(General formula for } t) \]

For our problem

\[ t = \frac{\ln \left(\frac{6000}{5000}\right)}{4 \ln \left(1 + \frac{.055}{4}\right)}. \]

Exercise 16 (Compound Interest): $12,000$ is invested in an account paying $3.85\%$, compounded monthly. How long will it take for the value of the account to reach $16,000$? Round your answer to the nearest hundredth of a year.

Solution. Given: $P = 12,000$, $F = 16,000$, $r = .0385$, and $n = 12$. Solve for $t$.

Apart from different numbers, this is just like Exercise 15. We use the same formula:

\[ t = \frac{\ln \left(\frac{16,000}{12,000}\right)}{12 \ln \left(1 + \frac{.0385}{12}\right)}. \]

Exercise 17 (Compound Interest): $3000$ is invested in an account paying $3.2\%$, compounded daily. How long will it take for the value of the account to reach $5000$? Round your answer to the nearest hundredth of a year.

Solution. Given: $P = 5000$, $F = 6000$, $r = .055$, and $n = 365$. Solve for $t$.

Apart from different numbers, this is just like Exercise 15. We use the same formula:

\[ t = \frac{\ln \left(\frac{5000}{3000}\right)}{365 \ln \left(1 + \frac{.055}{365}\right)}. \]

Exercise 18 (Compound Interest): $6500$ is invested in an account paying $5.15\%$, compounded continuously. How long will it take for the value of the account to reach $10,000$? Round your answer to the nearest hundredth of a year.

Solution. Given: Compounded continuously, $P = 6500$, $F = 10,000$, $r = .0515$. Find $t$.

Recall that $F = Pe^{rt}$. Solve for $t$:

\[ e^{rt} = \frac{F}{P} \quad \Rightarrow \quad \ln(e^{rt}) = \ln \left(\frac{F}{P}\right) \quad \text{(take } \log_e \text{ of both sides)} \]

\[ \Rightarrow \quad rt = \ln \left(\frac{F}{P}\right) \quad \text{(by property of logarithms)} \]

\[ \Rightarrow \quad t = \frac{1}{r} \ln \left(\frac{F}{P}\right) \]
For our problem, we want
\[ t = \frac{1}{.0515} \ln \left( \frac{10,000}{6500} \right) = 8.36 \text{ years} \]

Exercise 19 (Compound Interest): A deposit account pays interest of \( r \), compounded continuously. If the effective annual yield is 3.91%, what is \( r \)?
Solution. Given: compounded continuously, \( y = .0391 \). Want \( r \).
\[ r = \ln(1 + y) = \ln(1 + .0391) = \ln(1.0391) \]

Exercise 20 (Compound Interest): A deposit account pays interest of \( r \), compounded continuously. If the effective annual yield is 7.12%, what is \( r \)?
Solution. Given: compounded continuously, \( y = .0712 \). Want \( r \).
\[ r = \ln(1 + y) = \ln(1 + .0712) = \ln(1.0712) \]

Exercise 21 (Compound Interest): Suppose money earns at an annual rate of 3.65%, compounded quarterly. What is the present value of a $30,000 payment 4 years from now?
Solution. Given: \( F = 30,000 \), \( r = .0365 \), \( n = 4 \) (compounded quarterly), and \( t = 4 \). Want \( P \).
\[ P = F \left( 1 + \frac{r}{n} \right)^{-nt} = 30,000 \left( 1 + \frac{.0365}{4} \right)^{-4(4)} \]

Exercise 22 (Compound Interest): Suppose money earns at an annual rate of 5%, compounded monthly. What is the present value of a $5000 payment 6 years from now?
Solution. Given: \( F = 5000 \), \( r = .05 \), \( n = 12 \) (compounded monthly), and \( t = 6 \). Want \( P \).
\[ P = F \left( 1 + \frac{r}{n} \right)^{-nt} = 5000 \left( 1 + \frac{.05}{12} \right)^{-12(6)} \]

Exercise 23 (Compound Interest): Suppose money earns at an annual rate of 4.5%, compounded daily. What is the present value of a $125,000 payment 3 years and 6 months from now?
Solution. Given: \( F = 125,000 \), \( r = .045 \), \( n = 365 \) (compounded quarterly), and \( t = 3.5 \). Want \( P \).
\[ P = F \left( 1 + \frac{r}{n} \right)^{-nt} = 125,000 \left( 1 + \frac{.045}{365} \right)^{-365(3.5)} \]

Exercise 24 (Compound Interest): Suppose money earns at an annual rate of 5.2%, compounded continuously. What is the present value of a $23,000 payment 5 years from now?
Solution. Given: Compounded continuously, \( F = 23,000 \), \( r = .052 \), and \( t = 5 \). Want \( P \).
\[ P = Fe^{-rt} = 23,000e^{-(.052)(5)} \]
Exercise 25 (Compound Interest): Suppose money earns at an annual rate of 7.15%, compounded monthly. What is the future value of $13,000 after 3 years?

Solution. Given: \( n = 12 \) (compounded monthly), \( P = 13,000 \), \( t = 3 \), and \( r = .0715 \). Find \( F \).

\[
F = P \left(1 + \frac{r}{n}\right)^{nt} = 13,000 \left(1 + \frac{.0715}{12}\right)^{12(3)}
\]

Exercise 26 (Compound Interest): Suppose money earns at an annual rate of 4.6%, compounded continuously. What is the future value of $2000 after 7 years and 6 months?

Solution. Given: Compounded continuously, \( P = 2000 \), \( r = .046 \), and \( t = 7.5 \). Find \( F \).

\[
F = Pe^{rt} = 2000e^{.046(7.5)}
\]

Exercise 27 (Compound Interest): What is the yearly ratio of future value to present value that corresponds to an annual rate of 7.6%, compounded continuously? Solution. Given: Compounded continuously, \( r = .076 \). We want a yearly ratio \( R \):

\[
R \equiv \frac{F}{P} = e^r.
\]

(For \( t = 1 \) year, \( F(t)/P = e^{rt} = e^r \).) For \( r = .076 \), we have \( R = e^{.076} \).

Exercise 28 (Compound Interest): What is the annual rate of interest \( r \), compounded continuously, that corresponds to a yearly ratio of future value to present value of 1.10? Solution. Given: compounded continuously, \( R = 1.10 \). Want \( r \).

\[
r = \ln(R) = \ln(1.10).
\]
Exercise 29 (Compound Interest): A mortgage of $120,000 is to be paid off in 360 equal monthly payments, beginning one month after the purchase date. The interest on the mortgage is 5.78%, compounded continuously.

(i) What is the amount of each payment?

(ii) What is the total amount of all the payments?

Solution. Given: Compounded continuously, \( P = 120,000, r = 0.0578 \). The payments will be \( F_1, \ldots, F_{360} \), where the \( i \)th payment, \( F_i \), will be made \( i \) months, or \( i/12 \) years from now.

(i) The present value of \( F_i \) is \( P_i = F_i e^{-r(i/12)} \). Thus

\[
P = \sum_{i=1}^{360} P_i = \sum_{i=1}^{360} F_i e^{-r(i/12)}
\]

Since all payments \( F_i \) are equal, we can call that amount \( F = F_1 = F_2 = \cdots = F_{36} \), and factor it out of the sum:

\[
F \sum_{i=1}^{360} e^{-r(i/12)} = F \sum_{i=1}^{360} e^{-r(i/12)}
\]

Next, factor out \( e^{-r/12} \) and re-order the sum with \( k = i - 1 \) to get a geometric series:

\[
= Fe^{-r/12} \sum_{k=0}^{359} \left[ e^{-r/12} \right]^k
\]

Then

\[
F = P \left( e^{-r/12} \left[ \frac{1 - e^{-30r}}{1 - e^{-r/12}} \right] \right)^{-1} = Pe^{r/12} \left[ \frac{1 - e^{-(r/12)}}{1 - e^{-30r}} \right].
\]

Finish by substituting in the values for the given parameters.

(ii) To find the total amount, multiply the value of one of the (equal) payments by the total number of payments: \( 360F \).
Exercise 30 (Compound Interest): A sofa costing $899.99 can be paid for in 24 equal monthly payments, beginning one month after the purchase date. The interest on the unpaid balance is 21.5%, compounded continuously.

(i) What is the amount of each payment?
(ii) What is the total amount of all the payments?

Solution. Given: Compounded continuously, \( P = 899.99, r = .215 \). The payments will be \( F_1, \ldots, F_{24} \), where the \( i \)th payment, \( F_i \), will be made \( i \) months, or \( i/12 \) years from now.

(i) We can modify the formula we used to find the solution to Problem 29:

\[
F = Pe^{r/12} \left[ \frac{1 - e^{-(r/12)}}{1 - e^{-2r}} \right].
\]

(ii) To find the total amount, multiply the value of one of the (equal) payments by the total number of payments: \( 24F \).